Condensed matter physics 2013
Solutions to exercise 10

Discussion: 26. Nov. 2013

Solution to problem 38

The Landau level filling factor ν in a two-dimensional electron gas (2DEG) is given, for example, by the ratio between the electron density n and the density of flux quanta $B/(h/e) = eB/h$: $\nu = \frac{nh}{eB}$. The longitudinal resistance R_{xx} at low magnetic fields has SdH minima at integer filling factors, because backscattering is reduced due to the formation of edge states. At higher fields extended zero-resistance intervals develop which makes it difficult to find the integer value. The corresponding absolute number can be found by counting from the quantized Hall resistance plateaus at higher fields with $R_{xy}^p = \frac{h}{e^2}\frac{1}{p}$. The minima are labeled in the inset of the figure. From this, one can extract for each integer ν_i the values $1/\nu_i$ and the corresponding magnetic field B_i. Plotting

$$\frac{1}{\nu_i} = \frac{eB_i}{nh}$$

one finds a linear relation between the two variables which contains only the electron density n as free parameter, which determines the slope of the curve. One finds $n = 2.5 \cdot 10^{15}\,\mathrm{m}^{-2}$.

The classical, i.e. low-field Hall resistance in two dimensions yields

$$R_{xy} = \frac{B}{ne},$$

which also contains the electron density as the only free parameter and yields the same result. This is not the case in general, since the Hall effect measures the complete electron density, while the SdH oscillations stem from only one subband, i.e. one has to be more careful if more than one wave function perpendicular to the 2DEG is populated, e.g. at high electron densities.

Solution to problem 39

We choose the indices such that $T_{ij} \equiv T_{i\to j}$ denotes the total transmission of electrons from contact i to contact j. Now we use the Landauer-Büttiker formula for N_i quantum channels in each contact i:

$$I_i = \frac{2e^2}{h}\left[(N_i - R_i)\mu_i - \sum_j T_{ji}\mu_j\right] \tag{1}$$

In our example we have only one channel in each contact, i.e. $N_i = 1$ for all i, and no backscattering inside the contacts, $R_i = 0$. The transmission coefficient for contact pairs connected directly by edge channels (directional!) are one, i.e. $T_{12} = T_{34} = T_{45} = T_{61} = 1$, while the transmission in the opposite direction is zero: $T_{21} = T_{43} = T_{54} = T_{16} = 0$. The transmission between pairs separated by the quantum point contact are given by $T_{23} = T_{56} = T$. The other edge channel scatters into the opposite lead on the same side of the constriction: $T_{2\to 6} = T_{5\to 3} = 1 - T = R$ since $R + T = 1$. The current flows from contact 1 with $I_1 = I$ into contact 4 with $I_4 = -I$ while the current in the voltage terminals is zero by definition, i.e. $I_2 = I_3 = I_5 = I_6 = 0$. With this we can write eqn. (1) as a 6 by 6 matrix:

$$\begin{pmatrix} I \\ 0 \\ 0 \\ -I \\ 0 \\ 0 \end{pmatrix} = \frac{2e^2}{h} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -T & 1 & 0 & -R & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -R & 0 & 0 & -T & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}.$$

The matrix relating the currents to the contact potentials is called conductance matrix. Its columns and rows are *not* linearly independent. We can choose one potential to be zero, e.g. $\mu_4 \equiv 0$, which means that we can drop column 4 from the matrix. In addition, we can drop row 4, because one of the currents is the negative value of the sum of all the others due to current or charge conservation, i.e. $I_i = -\sum_{j \neq i} I_j$. Here we have $I_1 = -I_4$. The remaining system of equations reads

$$\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{2e^2}{h} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -T & 1 & -R & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -R & 0 & -T & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix}.$$

For more complex systems one can invert the resulting matrix (here 5×5), which yields the potentials in the contacts as a function of the currents given from the boundary conditions and the resistances can be calculated. In our simple example, we can solve the system of equations "by hand": The first equation (or line of the matrix) reads

$$I = \frac{2e^2}{h}(\mu_1 - \mu_6)$$

The second, $0 = -\mu_1 + \mu_2$, directly yields

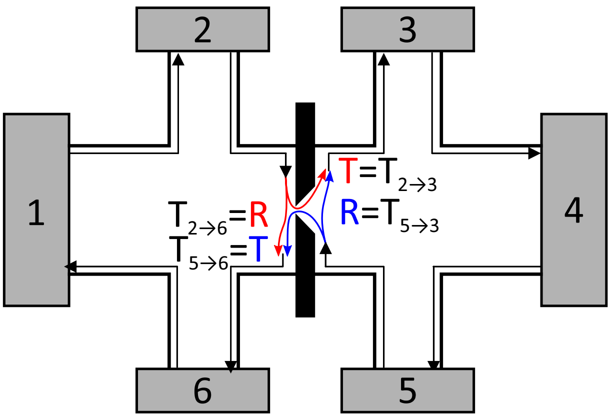
$$\mu_1 = \mu_2.$$

The third requires $0 = -T\mu_2 + \mu_3 - R\mu_5$, which results in $\mu_3 = T\mu_2 + R\mu_5$. The fourth line simply gives

$$\mu_5 = 0,$$

which, using the previous equations, leads to

$$\mu_3 = T\mu_1.$$



The fifth equation, $0 = -R\mu_2 - T\mu_5 + \mu_6$, yields

$$\mu_6 = R\mu_1.$$

Inserting these results into the first equation, one obtains

$$I = \frac{2e^2}{h}(\mu_1 - R\mu_6) = \frac{2e^2}{h}\mu_1(1-R) = \frac{2e^2}{h}\mu_1 T.$$

Now we can calculate the required resistances:

$$R_{14,23} = \frac{\mu_2 - \mu_3}{I_{14}} = \frac{\mu_1 - T\mu_1}{I} = \frac{h}{2e^2}\frac{1-T}{T}$$

and

$$R_{14,63} = \frac{\mu_6 - \mu_3}{I_{14}} = \frac{R\mu_1 - T\mu_1}{I} = \frac{h}{2e^2}\frac{R-T}{T} = \frac{h}{2e^2}\frac{1-2T}{T}.$$

Solution to problem 40

All N electrons in the area L^2 have to occupy the lowest Landau level. In turn, this level has to accommodate all N electrons, so that the level degeneracy, g, has to be larger than N. The degeneracy of every Landau level is given by the ratio between the magnetic flux through the system and the flux quantum (intuitively: every Landau state occupies approximately an area of $\pi\ell_B^2 = \frac{h}{2eB}$ with $\ell_B = \frac{\hbar}{eB}$ the magnetic length, see script p.4.53. An additional factor of 2 enters by the spin degeneracy. The total area contains $\frac{L^2}{2\pi\ell_B^2} = \frac{L^2 B}{h/e} = \frac{\Phi}{\Phi_0}$ states.) From this follows

$$g = \frac{\Phi}{\Phi_0} = \frac{L^2 B}{h/e} = \frac{eB}{h}L^2.$$

Requiring that $g > N$ one finds

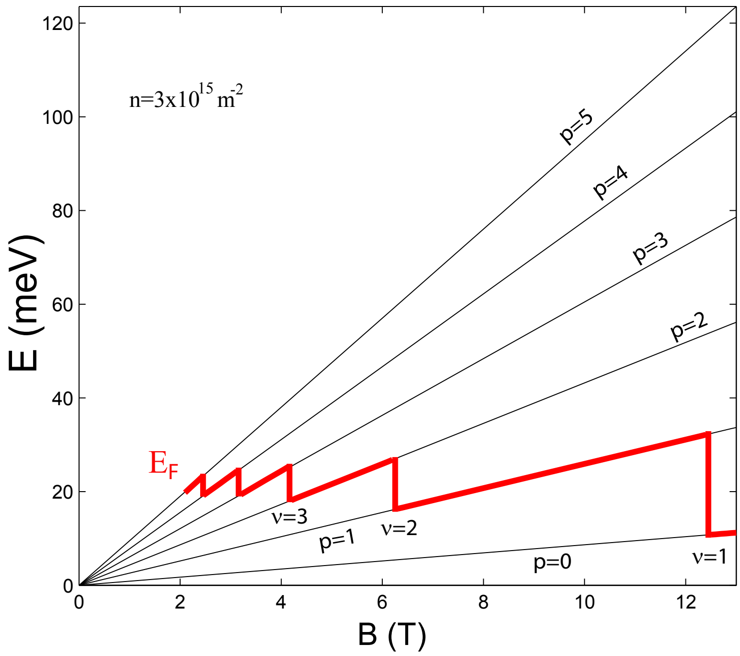
$$B > \frac{h}{eL^2}N.$$

For a fixed electron number N in a sample (no reservoir couples to the 2DEG), the Fermi energy is not fixed (see statistical mechanics later for a deeper understanding). If we consider only the kinetic energy of the electrons (e.g. we neglect the Zeemann energy), we find

$$E = \hbar\omega_c\left(p + \frac{1}{2}\right) = \frac{\hbar e B}{m}\left(p + \frac{1}{2}\right) \propto B$$

with $p \in \mathbb{N}_0$. This means that the energy of the states with quantum number p (Landau levels) raises linear in B with a constant of proportionality that depends on p. This leads to the "Landau Fan" depicted in the figure.

The Fermi energy E_F here is the highest energy of the occupied states (in thermal equilibrium). It is easiest to start at high fields with all electrons in the lowest Landau level ($p = 0$). When B is lowered, the energy is also lowered, but so is the degeneracy. When the condition $B > \frac{h}{eL^2}N$ is not met anymore, the next level has to be occupied and the Fermi energy jumps up to the $p = 1$ level. This happens at $B = \frac{h}{eL^2}N = \frac{nh}{e}$ or at filling factor $\nu = 1$ (for spin degenerate levels). Similar jumps in the Fermi energy occur at the next integer filling factors at lower fields (or even integer fillings if the spin degeneracy is not resolved anymore). This is depicted in the figure below for the spin degeneracy fully lifted and for an electron density of $n = 3 \times 10^{15}\,\mathrm{m}^{-2}$. N.b.: The figure is plotted to scale and it becomes clear that the jumps do not end at the same energies.



n=3x10^15 m^-2
E (meV)
B (T)
E_F
p=5
p=4
p=3
p=2
p=1
p=0
ν=3
ν=2
ν=1
0
20
40
60
80
100
120
0
2
4
6
8
10
12