Quantum Transport - Superconductivity I

- history:
  - discovery 1911 by Heike Kammerling-Onnes Leiden, RCT) for mercury showed $T_c \simeq 4.1K$
  - Meissner effect 1933:
    expulsion of magnetic field from the bulk of a superconductor sample
  - Theory:
    - London equations (1935)
    - Ginzburg Landau theory (1950)
      Landau theory of phase transitions applied to superconductivity
    - BCS (Bardeen Cooper Schriefer) theory (1957)
      first microscopic theory
    - Gorkov (1959): derivation of GL theory from BCS
  - 1962: prediction of the Josephson effect
    (B.D. Josephson)

London theory:

n_s : density of superconducting charge carriers

$$\vec{F} = e \cdot \left[\vec{E} + \cancel{\vec{v} \times \vec{B}} \right]$$

$$\Rightarrow \quad m \cdot \dot{\vec{v}} = e \cdot \vec{E} \quad \Big| \cdot e \cdot n_s$$

$$m \cdot \underbrace{e \cdot n_s \, \dot{\vec{v}}}_{\dot{\vec{j}}_s} = e^2 \cdot n_s \cdot \vec{E}$$

$\boxed{\text{1st London equation}}$

$$\boxed{\dot{\vec{j}}_s = \frac{e^2 \cdot n_s}{m} \cdot \vec{E}}$$

$$\vec{\nabla} \times \dot{\vec{j}}_s = \frac{e^2 n_s}{m} \cdot \vec{\nabla} \times \vec{E} \underset{\substack{\uparrow \\ \vec{\nabla} \times \vec{E} = -\dot{\vec{B}}}}{=} -\frac{e^2 n_s}{m} \cdot \dot{\vec{B}}$$

$$\Rightarrow \frac{\partial}{\partial t} \Big(\underbrace{\vec{\nabla} \times \vec{j}_s + \frac{e^2 n_s}{m} \cdot \vec{B}}_{=0} \Big) = 0$$

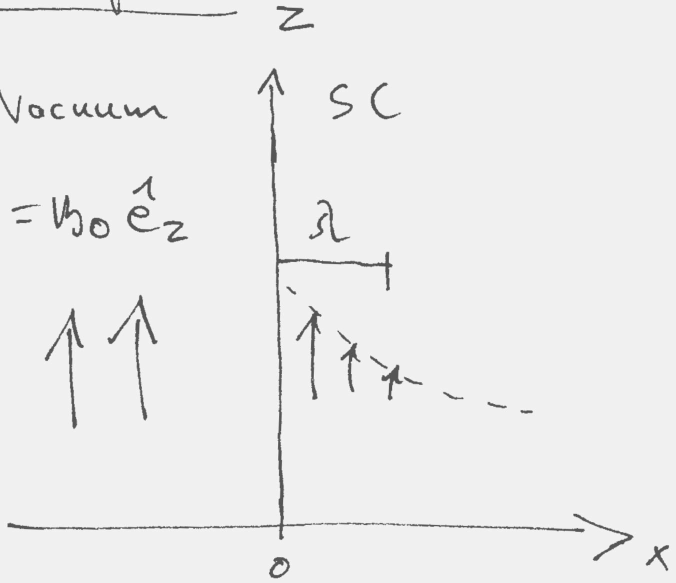
$\boxed{\text{2nd London equation}}$

$$\boxed{\vec{\nabla} \times \vec{j}_s = -\frac{e^2 n_s}{m} \cdot \vec{B}}$$

example :

Vacuum

$\vec{B} = B_0 \hat{e}_z$



Ampere's law : $\vec{\nabla} \times \vec{B} = \mu_0 \cdot \vec{j}$

$$\Rightarrow \vec{\nabla} \times \vec{j}_s = \frac{1}{\mu_0} \cdot \vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right)$$

$$= \frac{1}{\mu_0} \left[\vec{\nabla} \underbrace{\left(\vec{\nabla} \cdot \vec{B} \right)}_{=0} - \Delta \vec{B} \right] = - \frac{1}{\mu_0} \Delta \vec{B}$$

$$\vec{\nabla} \times \vec{j}_s = - \frac{1}{\mu_0} \Delta \vec{B} = - \frac{e^2 n_s}{m} \cdot \vec{B}$$

$$\Rightarrow \Delta \vec{B} = \frac{\mu_0 e^2 n_s}{m} \cdot \vec{B} = \frac{1}{\lambda^2} \cdot \vec{B}$$

$$\boxed{\lambda = \sqrt{\frac{m}{\mu_0 e^2 n_s}}}$$ London penetration depth

above example : $\vec{B}(x) = B_0 \cdot \hat{e}_z \cdot e^{-x/\lambda}$

$\rightarrow$ Meissner effect

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remark : T-dependence of λ : (from GL / BCS)

$$\lambda(T) = \frac{\lambda_{T=0}}{\sqrt{1 - \left(\frac{T}{T_c}\right)^4}} \rightarrow \infty , \ T \lesssim T_c$$

London theory can explain Meissner effect but not zero resistance -> BCS theory

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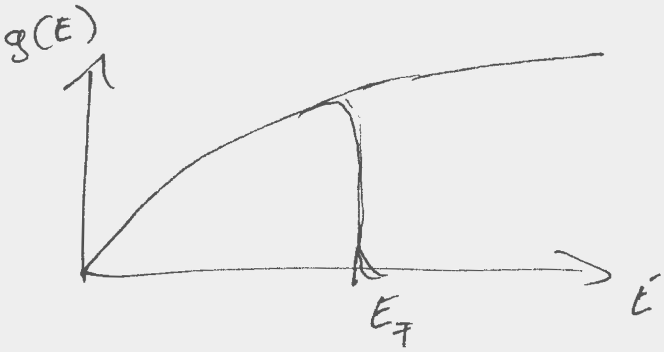
- free Fermi gas : noninteracting electrons, obey Fermi statistics
- in the presence of Coulomb repulsion : Fermi Liquid (qualitatively the same, "quasi-particles" instead of free electrons)
- if in addition to Coulomb repulsion there is some attractive interaction -> superconducting transition, new groundstate with paired electrons "Cooper pairs", excitation gap Δ



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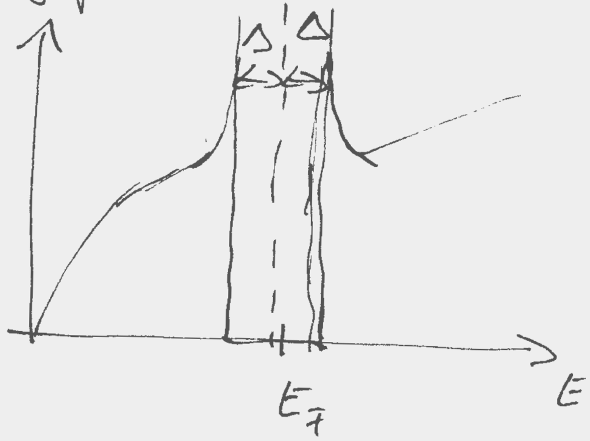


$$g_s(E) \propto \frac{|E-E_F|}{\sqrt{|E-E_F|^2-|\Delta|^2}}$$

mechanism for attractive interaction :

phonon exchange

Cooper pairs condense into a collective groundstate, which is characterized by a complex order parameter $\boxed{\Delta(\vec{r}) = |\Delta(\vec{r})| \cdot e^{i\varphi(\vec{r})}}$

"pair potential", macroscopic wave function

can also be given as $\Delta(\vec{r}) = \mathcal{V}(\vec{r}) \cdot F(\vec{r})$

attractive interaction parameter

pair amplitude

$F(\vec{r})$ is defined in second quantization:

$$F_{\sigma\sigma'}(\vec{r}) = \langle \Psi_\sigma(\vec{r}) \, \Psi_{\sigma'}(\vec{r}') \rangle$$

fermionic operators, anti-commute!

usually: $F_{\sigma\sigma'}(\vec{r}) = F_{\uparrow\downarrow}(\vec{r}) = -F_{\downarrow\uparrow}(\vec{r}) = F(\vec{r})$

$\rightarrow$ $|\Delta(\vec{r})|$ is rotationally invariant

$\rightarrow$ s-wave SC ($\uparrow\downarrow$)

but there are also p-wave SC ($\uparrow\uparrow$)

d-wave SC ($\uparrow\downarrow$, high T_c!)

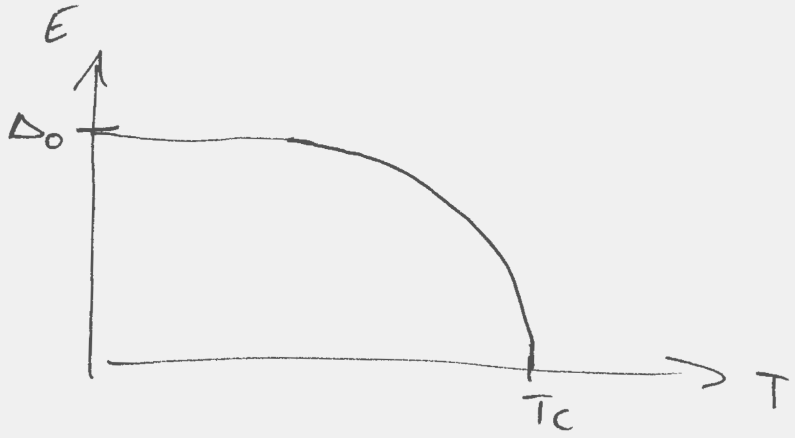
and many more

in this lecture: s-wave superconductors

temperature dependence for Δ from BCS:

$$\Delta(T) = \Delta(0) \cdot 1{,}74 \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}}$$

$$\Delta(0) = 1{,}76 \cdot k_B \cdot T_c$$



Δ is suppressed by T

($\rightarrow$ also by B for $B \rightarrow B_c$)

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important parameter: coherence length ξ_0

$$\boxed{\xi_0 = \frac{\hbar v_F}{\pi |\Delta|}}$$

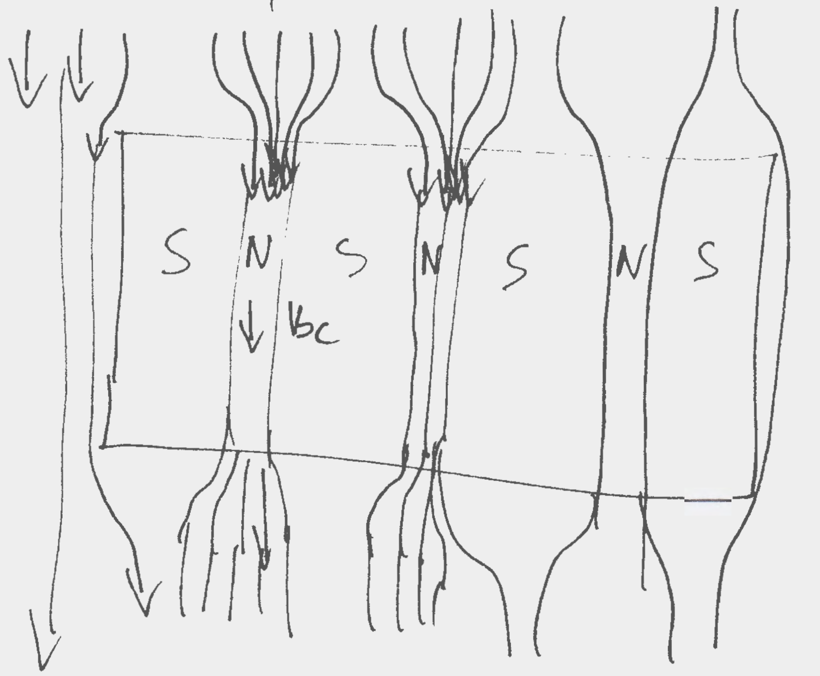
length scale of changes in $\Delta(\vec{r})$

($l_{ee} \gg \xi_0$)

note: diffusive case ($l_{ee} \ll \xi_0$):

$$\xi_0 = \sqrt{\frac{\hbar D}{2\Delta}}$$

breakdown of superconductivity with B:



partial penetration of B into S:

N-S interfaces inside superconductor

interface energy σ_{NS}

$$\sigma_{NS} \simeq \frac{B_c^2}{2\mu_0}(\xi - \lambda)$$

two cases:

$\sigma_{NS} > 0$: NS-interface "costs energy"
$(\xi > \lambda)$
-> no flux penetration
at $B = B_c$: all of S turns to N

$\boxed{\text{type I}}$

$\sigma_{NS} < 0$: NS-interfaces "reduce the total energy"
$(\xi < \lambda)$
-> flux penetrates S in form of "vortices"

$\boxed{\text{type II}}$

Ginzburg Landau parameter $\kappa = \frac{\lambda}{\xi}$

$\kappa < \frac{1}{\sqrt{2}}$: type I

$\kappa > \frac{1}{\sqrt{2}}$: type II

Josephson effect (after Feynman)

given: two superconductors, order parameters Ψ_1, Ψ_2

$$\Psi_1 = \sqrt{n_1}\, e^{i\varphi_1}$$

$$\Psi_2 = \sqrt{n_2}\, e^{i\varphi_2}$$

$n_{1/2}$: density of Cooper pairs

weak coupling K between Ψ_1 and Ψ_2:

energy of each individual SC : $U_{1/2}$

$$\left[\hat{E}\,\Psi = i\hbar \frac{\partial \Psi}{\partial t}\right] \Rightarrow \quad i\hbar \frac{\partial \Psi_1}{\partial t} = U_1 \Psi_1 - K\,\Psi_2$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = U_2 \Psi_2 - K\,\Psi_1$$

potential difference between 1 and 2: $U_1 - U_2 = q \cdot V$

set $\frac{U_1 - U_2}{2} = 0$

$$\Rightarrow \quad i\hbar \frac{\partial \Psi_1}{\partial t} = \frac{qV}{2}\Psi_1 - K\,\Psi_2$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = -\frac{qV}{2}\Psi_2 - K\,\Psi_1$$

with $\Psi_{1/2} = \sqrt{n_{1/2}} \cdot e^{i\varphi_{1/2}}$:

$$\frac{\partial \Psi_1}{\partial t} = \frac{1}{2\sqrt{n_1}} e^{i\varphi_1} \frac{dn_1}{dt} + i\sqrt{n_1} e^{i\varphi_1} \frac{d\varphi_1}{dt}$$

$$= \frac{qV}{2i\hbar} \sqrt{n_1}\, e^{i\varphi_1} - \frac{K}{i\hbar} \sqrt{n_2}\, e^{i\varphi_2} \quad \Big| \cdot e^{-i\varphi_1}$$

$$\frac{\partial \Psi_2}{\partial t} = \frac{1}{2\sqrt{n_2}} e^{i\varphi_2} \frac{dn_2}{dt} + i\sqrt{n_2} e^{i\varphi_2} \frac{d\varphi_2}{dt} = -\frac{qV}{2i\hbar} \sqrt{n_2}\, e^{i\varphi_2} - \frac{K}{i\hbar} \sqrt{n_1}\, e^{i\varphi_1}$$

$$\Big| \cdot e^{-i\varphi_2}$$

$$\frac{1}{2\sqrt{n_1}} \frac{dn_1}{dt} + i\sqrt{n_1} \frac{d\varphi_1}{dt} = \frac{qV}{2i\hbar} \sqrt{n_1} - \frac{K}{i\hbar} \sqrt{n_2}\, e^{+i(\varphi_2-\varphi_1)}$$

$$\frac{1}{2\sqrt{n_2}} \frac{dn_2}{dt} + i\sqrt{n_2} \frac{d\varphi_2}{dt} = -\frac{qV}{2i\hbar} \sqrt{n_2} - \frac{K}{i\hbar} \sqrt{n_1}\, e^{-i(\varphi_2-\varphi_1)}$$

real parts: $\dot{n}_1 = -2K\sqrt{n_1 n_2} \cdot \sin\varphi$

$\dot{n}_2 = 2K\sqrt{n_1 n_2}\, \sin\varphi$ $\quad (\varphi = \varphi_2 - \varphi_1)$

set $j = -q \cdot \frac{dn_1}{dt}$: $j = j_c \cdot \sin\varphi$, $j_c = 2qK\sqrt{n_1 n_2}$

$I = j \cdot A \Rightarrow \boxed{I = I_c \cdot \sin\varphi}$ 1st Josephson equation

$I_c = j_c \cdot A$

DC - Josephson effect

imaginary parts:

$$\frac{d\varphi_1}{dt} = -\frac{qV}{2\hbar} + K \cdot \sqrt{\frac{n_2}{n_1}} \cdot \cos\varphi$$

$$\frac{d\varphi_2}{dt} = \frac{qV}{2\hbar} + K \sqrt{\frac{n_1}{n_2}} \cdot \cos\varphi$$

assume $n_1 \simeq n_2$: $\quad \frac{d}{dt}(\varphi_2 - \varphi_1) = \frac{d\varphi}{dt} = \frac{qV}{\hbar}$

$$\boxed{\frac{d\varphi}{dt} = \frac{2e \cdot V}{\hbar}}$$

2nd Josephson equation

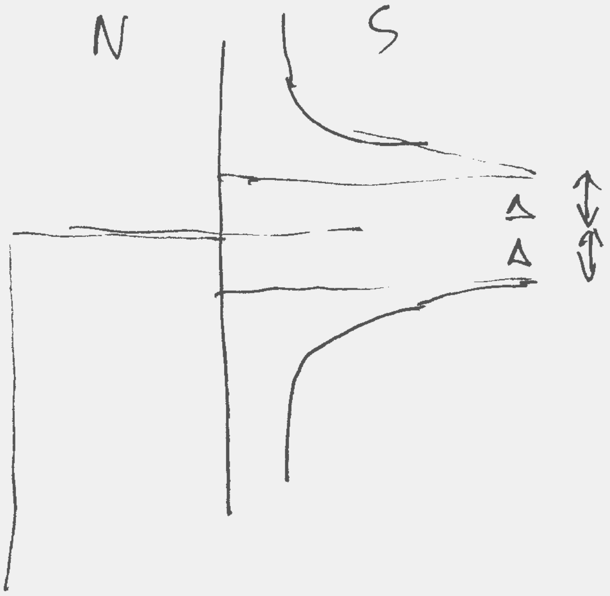
AC-Josephson effect

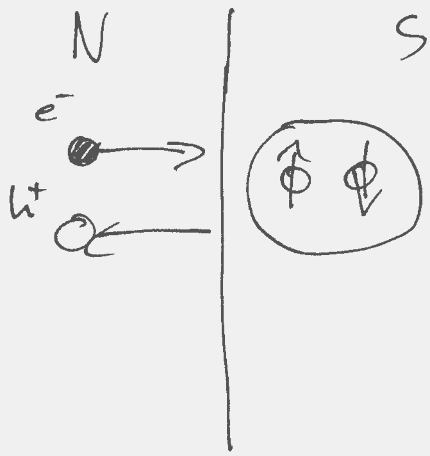
for $V = \text{const}$: $\quad \varphi = \varphi(0) + \frac{2eV}{\hbar} \cdot t$

$$\Rightarrow \quad j = j_c \cdot \sin\left(\varphi(0) + \frac{2eV}{\hbar} \cdot t\right)$$

AC-current

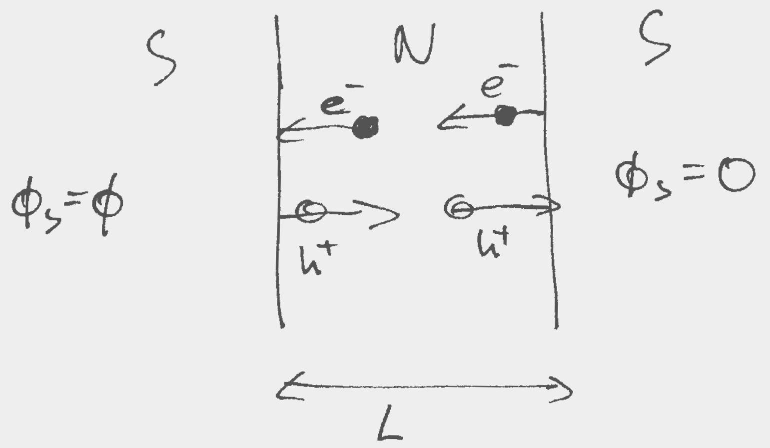
N-S interface: Andreev reflexion





in small S - N - S junctions : 11

Andreev bound states



-> consecutive Andreev reflexions at opposite S - N interfaces

-> in case of constructive interference between charge carrier trajectories : "Andreev bound state"

phase acquired during $e^- \to h^+$ Andreev reflexion : (without proof)

$$\varphi_{eh} = -\arccos\left(\frac{E}{\Delta}\right) + \phi_S \qquad ; \quad (E = E' - E_F)$$

inverse process :

$$\varphi_{he} = -\arccos\left(\frac{E}{\Delta}\right) - \phi_S$$

=> total phase : $\phi^{(1)}_{tot} = (k_h + k_e)\cdot L + \phi + 2\arccos\left(\frac{E}{\Delta}\right)$

dynamical phase : $(k_h + k_e)\cdot L \simeq 2\,EL/v_F$

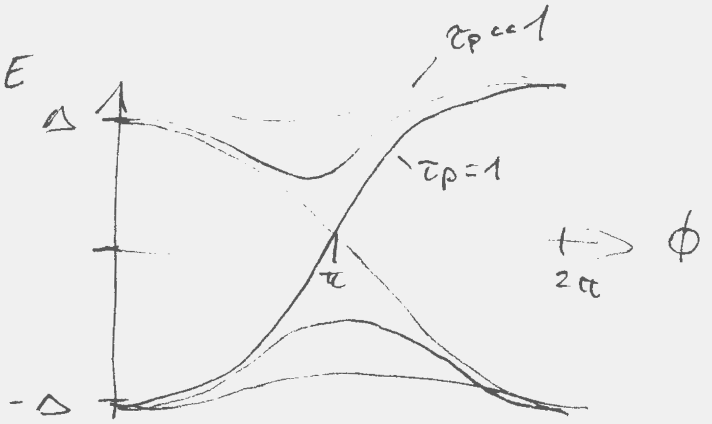
opposite process : $\phi^{(2)}_{tot} = (k_h + k_e)\cdot L - \phi + 2\arccos\left(\frac{E}{\Delta}\right)$

condition for a bound state:

$$\phi_{tot}^{(1)/(2)} = 2\pi \cdot n \quad ; n \in \mathbb{Z}$$

assume short junction ($L \ll \xi_0$)

$$\Rightarrow \quad E_\pm^{ABS} = \pm \Delta \cdot \cos\left(\phi/2\right)$$



for an interface with finite transmission ($\tau_p < 1$):

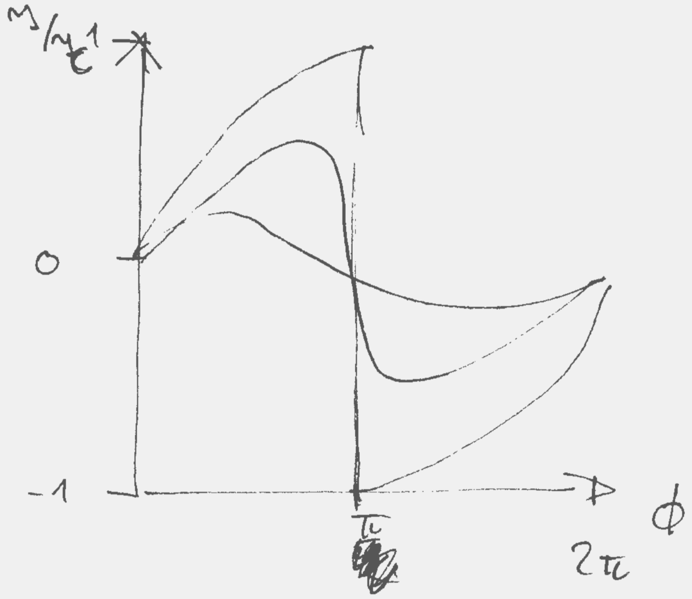
$$E_\pm^{ABS} = \pm \Delta \left[1 - \tau_p \cdot \sin^2\left(\frac{\phi}{2}\right)\right]$$

transfer of Cooper pairs through ABS → super current

$$I_s = \frac{2e}{\hbar} \sum_{+/-} \frac{\partial E_\pm^{ABS}(\phi)}{\partial \phi} \cdot \tanh\left(\frac{E_\pm^{ABS}(\phi)}{2k_BT}\right)$$

$$= \frac{e\Delta^2}{2\hbar} \cdot \frac{\tau \cdot \sin(\phi)}{E_+^{ABS}} \cdot \tanh\left(\frac{E_+^{ABS}(\phi)}{2k_BT}\right)$$

for $\tau_p \ll 1$: → recover 1st Josephson equation



for $\tau_p \ll 1$: $E_+^{ABS} \simeq \Delta$

→ summation over all channels p gives

$$I_s = \frac{\pi \cdot \Delta \cdot \sin(\phi)}{2e \cdot R_N} \cdot \tanh\left(\frac{\Delta}{k_B T}\right) \quad , R_N^{-1} = 2\frac{e^2}{h}\sum_p \tau_p$$

or $$I_c \cdot R_N = \frac{\pi \cdot \Delta}{2e} \quad (k_B T \ll \Delta)$$

Ambegaokar-Baratoff relation