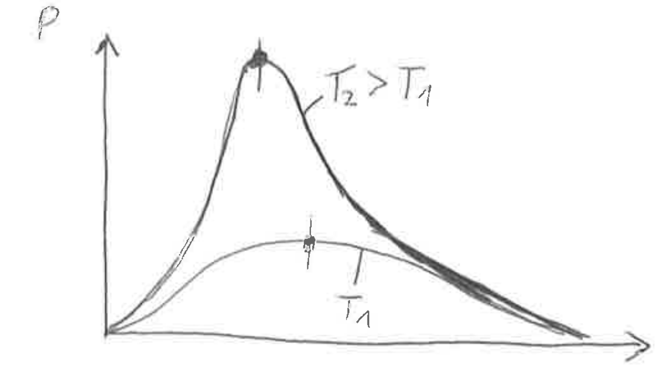
Solutions Test Exam (Tutorial Sheet 14)

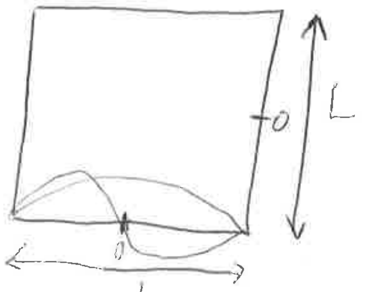
A1)



λ (attention: energy gets larger for smaller λ)

A2)
- plane wave $\Rightarrow A \cdot e^{ikx}$ with $\frac{(\hbar k)^2}{2m} = E$ or $k = \pm \frac{\sqrt{2mE}}{\hbar}$
- degeneracy $= 2$ $(\pm k)$
- $p = \pm \hbar k$, $v = p/m = \pm \hbar k / m$

A3)



Formally (not really necessary):

$$H\Psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\right)\Psi(x,y) = E \cdot \Psi(x,y)$$

inside square: $\phi = 0$
(outside : $\phi \to \infty$)

separation of variables: $\Psi(x,y) = \varphi_x \cdot \varphi_y \Rightarrow -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \varphi_x}{\partial x^2}\right) \cdot \varphi_y - \frac{\hbar^2}{2m}\varphi_x \cdot \frac{\partial^2 \varphi_y}{\partial y^2} = E \varphi_x \varphi_y$

divide by $\varphi_x \cdot \varphi_y$: $\underbrace{-\frac{\hbar^2}{2m} \cdot \frac{1}{\varphi_x} \cdot \frac{\partial^2 \varphi_x}{\partial x^2}}_{\text{depends only on } x} \underbrace{- \frac{\hbar^2}{2m} \frac{1}{\varphi_y} \cdot \frac{\partial^2 \varphi_y}{\partial y^2}}_{\text{depends only on } y} - \underset{\text{const.}}{E} = 0$

$\Rightarrow$ each term has to be constant

$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{\varphi_x} \cdot \frac{\partial^2 \varphi_x}{\partial x^2} \overset{!}{=} E_x \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 \varphi_x}{\partial x^2} = E_x \cdot \varphi_x \Rightarrow$ plane wave that fits to L!
(similar for φ_y)

~~Bis~~ Up to here one could also have guessed the solution.

$\Rightarrow$ Groundstate : $\varphi_x = \underset{\uparrow}{A} \cdot \cos\left(x \cdot \frac{\pi}{L}\right) \Rightarrow E_x^{(g)} = \frac{(\hbar k_x)^2}{2m} \equiv \frac{\left(\hbar \frac{\pi}{L}\right)^2}{2m}$

norm

$\varphi_y = \overset{\downarrow}{B} \cdot \cos\left(y \cdot \frac{\pi}{L}\right) \Rightarrow E_y^{(g)} = \frac{(\hbar k_y)^2}{2m} \equiv \frac{\left(\hbar \frac{\pi}{L}\right)^2}{2m}$

next higher energy: $\varphi_x = A \cdot \sin\left(x \cdot \frac{2\pi}{L}\right) \Rightarrow E_x^{(e)} = \frac{(\hbar k_x)^2}{2m} \equiv \frac{\left(\hbar \frac{2\pi}{L}\right)^2}{2m} = \frac{4 \cdot \left(\hbar \frac{\pi}{L}\right)^2}{2m}$

similarly $\quad E_y^{(e)} = \frac{4\left(\hbar \frac{\pi}{L}\right)^2}{2m}$

next higher energy: $E_x^{(n)} \equiv E_y^{(n)} = 9 \frac{(\hbar \pi/L)^2}{2m} \ldots$

$E_0 = E_x^{(g)} + E_y^{(g)} = 2 \cdot \frac{\left(\hbar \frac{\pi}{L}\right)^2}{2m}$ degeneracy 1

$E_1 = \underbrace{E_x^{(g)} + E_y^{(e)} = E_x^{(e)} + E_y^{(g)}}_{2 \text{ states}} = 5 \cdot \frac{\left(\hbar \frac{\pi}{L}\right)^2}{2m}$, degeneracy 2

$E_3 = \underbrace{E_x^{(e)} + E_y^{(e)}}_{\text{only } 1} = 8 \frac{(\hbar \pi/L)^2}{2m}$, degeneracy 1

A4) $e^{-\frac{2L \cdot \sqrt{2m(V_0 - E)}}{\hbar}} \equiv e^{-\frac{L}{L_0}} \Rightarrow L_0 = \frac{\hbar}{2\sqrt{2m(V_0 - E)}}$

$I \propto T$, $\underline{\underline{\frac{T_{low}}{T_{high}}}} = \frac{\cancel{16 \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}}{\cancel{16 \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}} \cdot \frac{e^{-\frac{L}{L_0}}}{e^{-\frac{L+\Delta L}{L_0}}} = \frac{\cancel{e^{-\frac{L}{L_0}}}}{\cancel{e^{-\frac{L}{L_0}}} \cdot e^{-\frac{\Delta L}{L_0}}} = \underline{\underline{e^{\frac{\Delta L}{L_0}}}}$

A5) i) $\Psi(t) = \frac{1}{\sqrt{3}} \cdot (\phi_0(t) + \phi_1(t) + \phi_2(t)) \underset{\uparrow}{=} \frac{1}{\sqrt{3}}\left(\phi_0 \cdot e^{-i\frac{E_0}{\hbar}t} + \phi_1 \cdot e^{-i\frac{E_1}{\hbar}t} + \phi_2 e^{-i\frac{E_2}{\hbar}t}\right)$

eigenstates!
$\rightarrow$ only a phase

$E_n = (n + 1/2)\hbar\omega$

$\Rightarrow \Psi(t) = \frac{1}{\sqrt{3}} \cdot \left(\phi_0 e^{-i(\cancel{0} + 1/2)\omega t} + \phi_1 e^{-i\frac{3\omega}{2}t} + \phi_2 e^{-i\frac{5}{2}\omega t}\right)$

A5) ii) $\langle E(t) \rangle = (\Psi(t), H\Psi(t))$

Scalar product

$$H\Psi(t) = \frac{1}{\sqrt{3}}\underbrace{(H\phi_0)}_{E_0\cdot\phi_0}\cdot e^{-i\frac{E_0}{\hbar}t} + \frac{1}{\sqrt{3}}\underbrace{(H\phi_1)}_{E_1\cdot\phi_1}e^{-i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{3}}\underbrace{(H\phi_2)}_{E_2\cdot\phi_2}\cdot e^{-i\frac{E_2}{\hbar}t}$$

Eigenstates form orthonormal basis $\Rightarrow (\phi_i, \phi_j) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

$$\Rightarrow \underline{\underline{\langle E(t)\rangle}} = \frac{1}{3}\left(e^{-i\frac{E_0}{\hbar}t}\phi_0 + e^{-i\frac{E_1}{\hbar}t}\phi_1 + e^{-i\frac{E_2}{\hbar}t}\phi_2,\ E_0\cdot e^{-i\frac{E_0}{\hbar}t}\phi_0 + E_1\cdot e^{-i\frac{E_1}{\hbar}t}\phi_1 + E_2 e^{-i\frac{E_2}{\hbar}t}\phi_2\right)$$

scalar product

$(\,,\,)$ is linear

$$= \frac{1}{3}\underbrace{\left(e^{-i\frac{E_0}{\hbar}t}\phi_0, E_0 e^{-i\frac{E_0}{\hbar}t}\phi_0\right)}_{\frac{1}{3}\underbrace{e^{+i\frac{E_0}{\hbar}t}\cdot e^{-i\frac{E_0}{\hbar}t}}_{=1}\cdot E_0\underbrace{(\phi_0,\phi_0)}_{=1}} + \frac{1}{3}\underbrace{\left(e^{-i\frac{E_0}{\hbar}t}\phi_0, E_1\cdot e^{-i\frac{E_1}{\hbar}t}\phi_1\right)}_{\text{all } (\phi_i,\phi_j)=0 \text{ for } i\neq j} + \ldots$$

$$+ \frac{1}{3}\underbrace{\left(e^{-i\frac{E_1}{\hbar}t}\phi_1, E_1 e^{-i\frac{E_1}{\hbar}t}\phi_1\right)}_{E_1} + \frac{1}{3}\underbrace{\left(e^{-i\frac{E_2}{\hbar}t}\phi_2, E_2 e^{-i\frac{E_2}{\hbar}t}\phi_2\right)}_{E_2} + (\text{zeros})$$

Könnte man auch direkt einfach hinschreiben

$$= \frac{1}{3}(E_0 + E_1 + E_2) = \frac{1}{3}\left[\tfrac{1}{2}\hbar\omega + \tfrac{3}{2}\hbar\omega + \tfrac{5}{2}\hbar\omega\right] = \underline{\underline{\frac{3}{2}\hbar\omega}}$$

$E_n = (n + \tfrac{1}{2})\hbar\omega$

iii) position operator expressed with ladder operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) = \frac{x_0}{\sqrt{2}}(a_+ + a_-) \quad \text{with} \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

a_+: upping, a_-: lowering

we know $(\phi_i, \phi_j) = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$ and $a_+\phi_n = \sqrt{n+1}\cdot\phi_{n+1}$

and $a_-\phi_n = \sqrt{n}\cdot\phi_{n-1}$

not necessary in exam (from lecture)

e.g. $a_+\phi_1 = \sqrt{2}\phi_2$; $a_+\phi_2 = \sqrt{3}\phi_3$; $a_+\phi_0 = \phi_1$; $a_-\phi_0 = 0$; $a_-\phi_1 = \phi_0$, $a_-\phi_2 = \sqrt{2}\phi_1$ ...

$$\Rightarrow \langle x \rangle = \frac{x_0}{\sqrt{2}} \cdot \left(\Psi, (a_+ + a_-) \Psi \right) \quad . \quad \text{with} \quad \omega_i = \frac{E_i}{\hbar} :$$

$$a_+ \Psi(t) = \frac{1}{\sqrt{3}} \left(\underbrace{a_+ \phi_0}_{\phi_1} e^{-i\omega_0 t} + \underbrace{a_+ \phi_1}_{\sqrt{2}\phi_2} e^{-i\omega_1 t} + \underbrace{a_+ \phi_2}_{\sqrt{3}\phi_3} e^{-i\omega_2 t} \right)$$

$$= \frac{1}{\sqrt{3}} \left(e^{-i\omega_0 t} \cdot \phi_1 + \sqrt{2} \cdot e^{-i\omega_1 t} \cdot \phi_2 + \sqrt{3} \cdot e^{-i\omega_2 t} \cdot \phi_3 \right)$$

$$a_- \Psi(t) = \frac{1}{\sqrt{3}} \left(\underbrace{a_- \phi_0}_{0} e^{-i\omega_0 t} + \underbrace{a_- \phi_1}_{\phi_0} e^{-i\omega_1 t} + \underbrace{a_- \phi_2}_{\sqrt{2}\phi_1} e^{-i\omega_2 t} \right)$$

$$= \frac{1}{\sqrt{3}} \left(e^{-i\omega_1 t} \cdot \phi_0 + \sqrt{2} \cdot e^{-i\omega_2 t} \cdot \phi_1 \right)$$

$$\Rightarrow (\Psi, a_+ \Psi) = \frac{1}{3} \left(\phi_0 e^{-i\omega_0 t} + \phi_1 \cdot e^{-i\omega_1 t} + \phi_2 e^{-i\omega_2 t} , e^{-i\omega_0 t} \phi_1 + \sqrt{2} e^{-i\omega_1 t} \phi_2 + \sqrt{3} e^{-i\omega_2 t} \cdot \phi_3 \right)$$

$\neq 0$ $\quad$ $\neq 0$ $\quad$ rest $= 0$

$$= \frac{1}{3} \left(e^{-i\omega_1 t} \phi_1 , e^{-i\omega_0 t} \phi_1 \right) + \frac{1}{3} \left(e^{-i\omega_2 t} \cdot \phi_2 , \sqrt{2} e^{-i\omega_1 t} \phi_2 \right)$$

$$= \frac{1}{3} e^{+i\omega_1 t} \cdot e^{-i\omega_0 t} \underbrace{(\phi_1, \phi_1)}_{=1} + \frac{\sqrt{2}}{3} e^{+i\omega_2 t} \cdot e^{-i\omega_1 t} \cdot \underbrace{(\phi_2, \phi_2)}_{=1}$$

$$= \frac{1}{3} e^{i(\omega_1 - \omega_0) t} + \frac{\sqrt{2}}{3} e^{i(\omega_2 - \omega_1) t} = \frac{1}{3} e^{i\omega t} + \frac{\sqrt{2}}{3} e^{i\omega t} = \frac{1}{3} e^{i\omega t} (1 + \sqrt{2})$$

$$\left.\begin{aligned} \omega_2 &= \frac{E_2}{\hbar} = \frac{(2 + 1/2)\hbar\omega}{\hbar} = 5/2\, \omega \\ \omega_1 &= \frac{E_1}{\hbar} = \frac{(1 + 1/2)\hbar\omega}{\hbar} = 3/2\, \omega \\ \omega_0 &= E_0/\hbar = \frac{(0 + 1/2)\hbar\omega}{\hbar} = 1/2\, \omega \end{aligned}\right\} \quad \begin{aligned} \omega_2 - \omega_1 &= \omega \\ \omega_1 - \omega_0 &= \omega \end{aligned}$$

$$\Rightarrow (\Psi, a_- \Psi) = \frac{1}{3} \left(\phi_0 e^{-i\omega_0 t} + \phi_1 e^{-i\omega_1 t} + \phi_2 e^{-i\omega_2 t} , e^{-i\omega_1 t} \cdot \phi_0 + \sqrt{2} e^{-i\omega_2 t} \cdot \phi_1 \right)$$

$\neq 0$ $\quad$ $\neq 0$

$$= \frac{1}{3} \left(e^{-i\omega_0 t} \phi_0 , e^{-i\omega_1 t} \phi_0 \right) + \frac{1}{3} \sqrt{2} \left(e^{-i\omega_1 t} \cdot \phi_1 , e^{-i\omega_2 t} \cdot \phi_1 \right)$$

$$= \frac{1}{3} e^{+i\omega_0 t} e^{-i\omega_1 t} \underbrace{(\phi_0, \phi_0)}_{=1} + \frac{\sqrt{2}}{3} e^{+i\omega_1 t} e^{-i\omega_2 t} \underbrace{(\phi_1, \phi_1)}_{=1}$$

$$= \frac{1}{3} e^{-i(\omega_1 - \omega_0) t} + \frac{\sqrt{2}}{3} e^{-i(\omega_2 - \omega_1) t} = \frac{1}{3} e^{-i\omega t} + \frac{\sqrt{2}}{3} e^{-i\omega t} = \frac{1}{3} (1 + \sqrt{2}) e^{-i\omega t}$$

$$\Rightarrow \underline{\langle x \rangle} = \frac{x_0}{\sqrt{2}} \cdot \frac{1}{3} (1 + \sqrt{2}) \cdot \left(e^{i\omega t} + e^{-i\omega t} \right) = \sqrt{2} (1 + \sqrt{2}) \cdot \frac{x_0}{3} \cdot \cos(\omega t)$$

A6) a) Error in posed problem! We <u>should have</u> $H = \frac{\hat{L}^2}{2mr^2}$ (sorry...)

We know: $\hat{L}^2 Y_\ell^m = \hbar^2 \ell(\ell+1) \cdot Y_\ell^m$

$\Rightarrow$ Eigenvalues: $\hbar^2 \ell(\ell+1)$ $\Rightarrow$ Eigenvalues of $\frac{\hat{L}^2}{2mr^2}$ ← just a number in this problem!

$\hookrightarrow \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$

Energies in units of $\frac{\hbar^2}{2mr^2}$: $\ell(\ell+1) = 0 \quad (\ell=0)$

$= 2 \quad (\ell=1)$

$= 6 \quad (\ell=2)$

...

For each ℓ we have $2\ell+1$ ℓ_z-values.

b) Without B-field only n ~~gives~~ is relevant for the energy

$\hookrightarrow E_n = \frac{1\,Ry}{n^2} = \frac{-13.6\,eV}{n^2}$

$E_3 = -\frac{13.6\,eV}{9}$; $E_2 = -\frac{13.6\,eV}{4}$ $\Rightarrow \Delta E = \hbar\omega_{light} = E_3 - E_2$ ~~= ...~~

$= -13.6 \cdot \left(\frac{1}{9} - \frac{1}{4}\right) = \underline{\underline{0.139\,Ry}}$

$\Rightarrow \underline{\underline{\lambda_{light}}} = \frac{c}{\nu} = \frac{c \cdot 2\pi}{\omega_{light}} = \frac{2\pi c \cdot \hbar}{\Delta E} = \underline{\underline{\frac{c \cdot h}{\Delta E}}}$

A7) Also these results one could guess easily without calculation.

Still:

a) after measurement: $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ („collapse of wave function")

possible measurement outcomes (see postulate 4)

$\sigma_x \begin{pmatrix} a \\ b \end{pmatrix} \stackrel{!}{=} S \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ $\Rightarrow \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

↑ eigenvalue

$\Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = S \cdot \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = S \cdot a$ and $a = S \cdot b = S^2 \cdot a \Rightarrow$ ~~a~~ $S^2 = 1 \Rightarrow \underline{\underline{S = \pm 1}}$

$\Rightarrow$ two possible outcomes: $\pm\hbar/2$ (Definition of Paulimatrices!)

What probability?

Eigenvectors to eigenvalues (postulate 4):

- $\sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \overset{!}{=} +\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = a$

  with normalization $a^2 + b^2 = 1 \Rightarrow 2a^2 = 1$ or $a = \frac{1}{\sqrt{2}}$

  $\Rightarrow$ eigenvector $s_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \overset{!}{=} -\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow b = -a$

  normalization $\Rightarrow s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ Probabilities: $\underline{\underline{W_{s=+1}}} = |(s_1, \Psi)|^2 = \left|\frac{1}{\sqrt{2}}(1,1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = \underline{\underline{\frac{1}{2}}}$ (postulate 4)

$\underline{\underline{W_{s=-1}}} = |(s_2, \Psi)|^2 = \left|\frac{1}{\sqrt{2}}(1,-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = \underline{\underline{\frac{1}{2}}}$

b) state s_1: $W_{up} = \left|(1,0) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}$

$W_{down} = \left|(0,1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}$

state s_2: $\underline{\underline{W_{up}}} = \left|(1,0) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right|^2 = \underline{\underline{\frac{1}{2}}}$

$\underline{\underline{W_{down}}} = \left|(0,1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right|^2 = \underline{\underline{\frac{1}{2}}}$

all this one can also guess from rotating the axes by 90°!

c) „error": the „first" measurement means the one in a)

$\Rightarrow$ the <u>very first</u> measurement projects the state to an eigenstate of σ_z.

Measuring again gives again the same eigenvalue +1 (or in $\underline{\underline{S_z : +\hbar/2}}$) with $\underline{\underline{\text{100\% certainty}}}$

($\Leftrightarrow$ the σ_x measurement rotates the state to x-axis...)