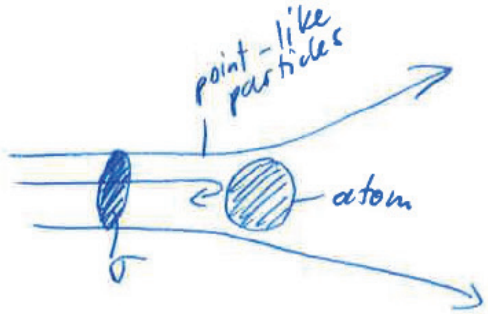
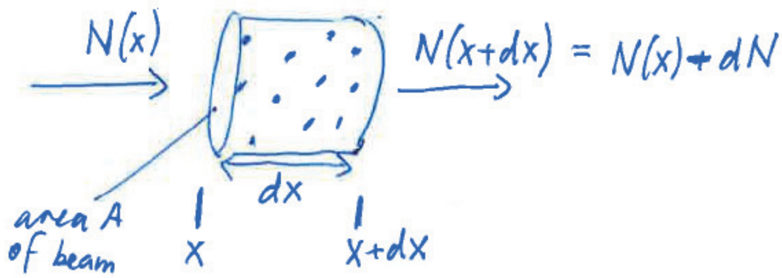
Solutions to Problem Sheet 1

1a) • scattering cross section for one atom



• Many independent (stationary) atoms



• probability for one atom in the beam to scatter in the volume (see above) $A \cdot dx$

$$p \overset{\text{independent atoms}}{\approx} \underbrace{M}_{\substack{\text{number}\\\text{of atoms}\\\text{in volume}}} \cdot \sigma / A = \underbrace{n}_{\text{density}} \cdot (A \cdot dx) \cdot \sigma / A = n\sigma \cdot dx$$

• number of scattered particles in beam: $dN = N(x) \cdot p$

[$\Rightarrow$ number of particles remaining in beam: $N(x+dx) = N(x) - dN$]

change of particle number in beam: $\underline{\underline{\frac{dN}{dx} = -N(x) \cdot n\sigma \equiv N'(x)}}$

b) from a) $\frac{dN}{N} = -n\sigma dx$

boundary condition: $N(x=0) \equiv N_0$, beam enters chamber at $x=0$

$\Rightarrow \int_{N_0}^{N(L)} \frac{dN}{N} = -n\sigma \int_0^L dx$; L: length of path in the gas

$\Rightarrow \ln\left(\frac{N(L)}{N_0}\right) = -n\sigma \cdot L \quad \Rightarrow \underline{\underline{N(L) = N_0 \cdot e^{-n\sigma L}}}$

1c) $p = 0.1\,\text{mbar}$, $T = 300\,K$, $L = 0.3\,m$, $\frac{N(L)}{N_0} = 0.3$

$pV = MK_BT \Rightarrow n = \frac{M}{V} = \frac{p}{K_BT}$

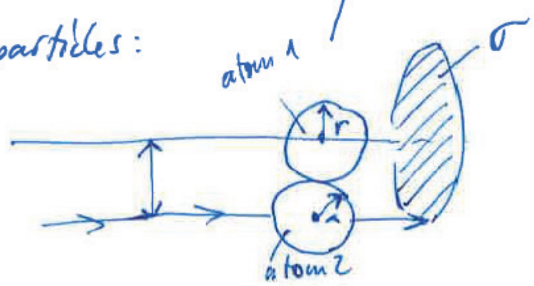
From b): solve for σ (start from second to last step :)

$$\underline{\underline{\sigma}} = -\frac{1}{nL}\cdot \ln\left(\frac{N(L)}{N_0}\right) = \underline{\underline{1.66\cdot 10^{-21}\,m^2}}$$

insert number above

radius of He-atom: $\sigma \approx (2r_{He})^2\pi \Rightarrow \underline{\underline{r_{He}}} \approx \sqrt{\frac{\sigma}{4\pi}} \approx 1.2\cdot 10^{-11}\,m = \underline{\underline{0.12\,\text{Å}}}$

not point-like particles:



2a) $M = 1\,kg$, $1\,mol \,\hat{=}\, 27.98\,g$ $(M_W = 27.98\,g/mol)$

$\Rightarrow$ number of required ^{28}Si atoms: $N = \frac{M}{M_W} = 35.7\,mol = \underline{\underline{2.15\cdot 10^{25}\text{ atoms}}}$

b) number of unit cells ("boxes") with edge length a (each contains $n = 8$ atoms)

$N_{uc} = \frac{N}{8}$, $V_{uc} = a^3$, volume of a sphere $V = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$

$$\Rightarrow \underline{\underline{V}} = N_{uc}\cdot V_{uc} = \frac{\overset{\text{from 2a)}}{N}}{8}a^3 = 4.31\cdot 10^{-4}\,m^3 = \underline{\underline{431\,cm^3}}$$

$$\underline{\underline{d}} = 2r = 2\cdot\sqrt[3]{\frac{3V}{4\pi}} = \underline{\underline{9.37\,cm}}$$

2c) relative uncertainty: $\frac{\Delta N_A}{N_A}$; $N_A = \frac{M_W\cdot n}{\rho\cdot a^3}$

given in text

i) with respect to M_W: $\frac{\Delta N_A}{N_A} = \frac{1}{N_A}\cdot\frac{\partial N_A}{\partial M_W}\cdot\Delta M_W = \frac{1}{N_A}\cdot\frac{n}{\rho a^3}\cdot\Delta M_W = \frac{\Delta M_W}{M_W} = \underline{\underline{6.8\cdot 10^{-11}}}$

ii) with respect to a: $\frac{\Delta N_A}{N_A} = \frac{1}{N_A}\frac{\partial N_A}{\partial a}\cdot\Delta a = \frac{1}{N_A}\cdot\frac{nM_W}{\rho}\left(\frac{-3}{a^4}\right)\cdot\Delta a = -3\frac{\Delta a}{a} = \underline{\underline{4.8\cdot 10^{-8}}}$

$\Rightarrow$ uncertainty in a dominates uncertainty in N_A

2d) Goal: $\frac{\Delta N_A}{N_A} < 10^{-8}$

Here, the number of atoms is constant, but some ^{28}Si are replaced by ^{29}Si with a molar mass of $\sim 29\ g/mol \Rightarrow \sim 1\ g/mol$ more than ^{28}Si

$\Rightarrow$ number of atoms: $\overset{1\,mol}{N_A}$, fraction of ^{29}Si: x $\Rightarrow$ additional mass $\Delta M = x \cdot 1g$

In the definition we would assume that this additional mass is due to more ^{28}Si atoms $\Rightarrow \Delta N_A = \frac{\Delta M}{M_W^{(28)}} = x \cdot \frac{1g}{28 g/mol} \cdot N_A$

$$\Rightarrow \frac{\Delta N_A}{N_A} < 10^{-8} = x \cdot \frac{1g}{28g} \Rightarrow \underline{\underline{x}} < 28 \cdot 10^{-8} = \underline{\underline{2.8 \cdot 10^{-7}}}$$

2e) Each vacancy reduces the mass by $28\ g/mol \Rightarrow \Delta M = x \cdot 28g$

Similar as in 2d: $\Delta N_A = \frac{\Delta M}{M_W^{(28)}} = x \cdot \frac{28\,g}{28 g/mol} \cdot N_A = x \cdot N_A$

$$\Rightarrow \frac{\Delta N_A}{N_A} = \underline{\underline{x \overset{!}{<} 10^{-8}}}$$

f) Here we consider the effect of a change in volume at constant mass, which leads to an (apparent) change in ρ.

$\Rightarrow$ mass M constant $\Rightarrow \rho_0 = \frac{M}{V_0}$; $\rho_{ox} = \frac{M}{V_{ox}}$; $V_{ox} = (r+\Delta r)^3 \frac{4}{3}\pi$; $V_0 = r^3 \frac{4}{3}\pi$

$$\Rightarrow N_A^{(ox)} = \frac{M_W n}{\rho_{ox} a^3} = \underbrace{\frac{M_W n}{\rho_0 a^3}}_{N_A} \cdot \frac{V_{ox}}{V_0} \Rightarrow \frac{\Delta N_A}{N_A} = \frac{N_A^{(ox)} - N_A}{N_A} = \frac{V_{ox}}{V_0} - 1 \overset{!}{<} 10^{-8}$$

$$\frac{V_{ox}}{V_0} = \frac{(r+\Delta r)^3}{r^3} \underset{\text{Pascal's triangle}}{=} 1 + \frac{3r^2\Delta r}{r^3} + \frac{3r\Delta r^2}{r^3} + (\Delta r/r)^3 \underset{r \gg \Delta r}{=} 1 + 3 \cdot \frac{\Delta r}{r} \Rightarrow \frac{3\Delta r}{r} \overset{!}{<} 10^{-8}$$

from 2b

$$\Rightarrow \underline{\underline{\Delta r}} < \frac{1}{3} \cdot 10^{-8} \cdot r \approx \underline{\underline{156\ pm}}$$

(similarly: $\frac{\Delta N_A}{N_A} = \frac{1}{N_A} \frac{\partial N_A}{\partial V} \cdot \Delta V = \ldots = \frac{\Delta V}{V} < 10^{-8}$)

2g) $\Delta r < 1.56 \cdot 10^{-10}\ m$ (from 2f)

$r \approx 4.685 \cdot 10^{-2}\ m$ (from 2b), $\alpha = 2.6 \cdot 10^{-6} \frac{m}{m \cdot K}$

$$\underline{\underline{\Delta T}} = \frac{\Delta r}{r \cdot \alpha} = \underline{\underline{1.28\ mK}}$$