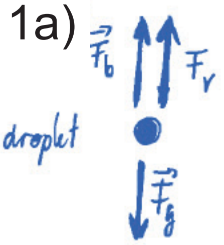
Solutions to Homework Problem Sheet 2

1a)



Forces: gravitation : $|\vec{F}_g| = mg = \rho_{oil}\cdot V_{drop}\cdot g = \rho_{oil}\cdot g\cdot\frac{4}{3}r^3\pi$

viscous drag : $|\vec{F}_v| = 6\pi\eta_{air}\cdot r\cdot v_t$

buoyancy (Auftrieb) : $|\vec{F}_b| = \rho_{air}\cdot V_{drop}\cdot g = \rho_{air}\, g\,\frac{4}{3}r^3\pi$

(direction of forces → intuition/graph)

terminal velocity ⇔ $F_g \overset{!}{=} F_b + F_v$

$$\rho_{oil}\, g\,\frac{4}{3}r^{\cancel{3}\,2}\cancel{\pi} \overset{!}{=} \rho_{air}\, g\,\frac{4}{3}r^{\cancel{3}\,2}\cancel{\pi} + 6\cancel{\pi}\,\eta_{air}\cdot\cancel{r}\,v_t \qquad \Big|\,\frac{1}{\pi r} + \text{some muddling}$$

$$r^2\cdot\frac{4}{3}g(\rho_{oil}-\rho_{air}) = 6\,\eta_{air}\,v_t$$

$$\underline{\underline{r = \sqrt{\frac{9\,\eta_{air}\cdot v_t}{2g(\rho_{oil}-\rho_{air})}}}}$$

b) charge of droplet : q

Forces : electric field: $|\vec{F}_E| = |q\cdot\vec{E}|$

gravitation : as above

buoyancy : as above

viscous drag : since $v=0 \Rightarrow F_v = 0$

$$\Rightarrow \quad |\vec{F}_E| + |\vec{F}_b| \overset{!}{=} |\vec{F}_g|$$

$$F_E = q\cdot\underset{\substack{\uparrow\\ \text{adjusted}\\ \text{in experiment}}}{|\vec{E}|} \overset{!}{=} \frac{4}{3}\pi\underset{\substack{\uparrow\\ \text{from a)}}}{r^3}g\cdot(\rho_{oil}-\rho_{air})$$

$$\Rightarrow \quad \underline{\underline{q}} = \frac{4}{3}\pi g(\rho_{oil}-\rho_{air})\frac{1}{E}\left(\frac{9\,\eta_{air}\,v_t}{2g(\rho_{oil}-\rho_{air})}\right)^{3/2} = \underline{\underline{\frac{18\pi}{E}\left(\frac{\eta_{air}^3\,v_t^3}{2(\rho_{oil}-\rho_{air})\cdot g}\right)^{1/2}}}$$

c) all observed charges are multiples of $e := 1{,}6\cdot 10^{-19}\,C$.
→ charge occurs quantized

2a)

$$\vec\nabla\wedge\left(\vec\nabla\wedge\vec E\right)\underset{\substack{\uparrow\\\text{given vector}\\\text{identity}}}{\overset{(i)}{=}}\vec\nabla\left(\vec\nabla\cdot\vec E\right)-\vec\nabla^2\vec E\underset{\substack{\uparrow\\\vec E=\frac{1}{\varepsilon}\vec D\\\vec\nabla\cdot\vec D=0}}{=}-\vec\nabla^2\vec E$$

$$\underset{\substack{\uparrow\\\vec\nabla\wedge\vec E=-\frac{\partial\vec B}{\partial t}}}{\overset{(ii)}{=}}-\vec\nabla\wedge\left(\frac{\partial\vec B}{dt}\right)\underset{\substack{\uparrow\\\frac{\partial}{\partial t}\text{ and}\\\text{spatial}\\\text{derivatives}\\\text{interchange}}}{=}-\frac{\partial}{\partial t}\left(\vec\nabla\wedge\vec B\right)\underset{\substack{\uparrow\\\vec B=\mu\vec H\\\vec\nabla\wedge\vec H=\frac{\partial\vec D}{dt}}}{=}-\frac{\partial}{\partial t}\left(\mu\frac{\partial\vec D}{\partial t}\right)\underset{\substack{\uparrow\\\vec D=\varepsilon\vec E}}{=}-\varepsilon\mu\frac{\partial^2}{dt}\vec E\underset{\substack{\uparrow\\\text{define:}}}{=}-\left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec E$$

define:

$c=(\varepsilon_0\mu_0)^{-1/2}$ (vacuum light speed)

$\tilde c=(\varepsilon\mu)^{-1/2}=:\frac{c}{n}$

(speed of light in medium)

n: refraction index

$$\Rightarrow\ \underline{\vec\nabla^2\vec E-\left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec E\overset{!}{=}0}$$

b) Ansatz: $\vec E(\vec x,t)=\vec E_0\cdot f(\overbrace{\vec k\cdot\vec x-\omega t}^{\varphi:=k_xx+k_yy+k_zz-\omega t})$, $\vec k$: constant, ω: constant

$$\vec\nabla^2\vec E=\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}^2\vec E=\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\vec E=\underset{\substack{\uparrow\\\text{constant!}}}{\vec E_0}\frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial\varphi}\cdot\underbrace{\frac{\partial\varphi}{\partial x}}_{k_x}\Big)+\vec E_0\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial\varphi}\underbrace{\frac{\partial\varphi}{\partial y}}_{k_y}\Big)+\vec E_0\frac{\partial}{\partial z}\Big(\frac{\partial f}{\partial\varphi}\cdot\underbrace{\frac{\partial\varphi}{\partial z}}_{k_z}\Big)$$

(3 time the same structure)

$$=\vec E_0\cdot\left(k_x\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial\varphi}\right]+k_y\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial\varphi}\right]+k_z\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial\varphi}\right]\right)$$

$$=\vec E_0\Big(k_x\frac{\partial^2f}{\partial\varphi^2}\cdot\underbrace{\frac{\partial\varphi}{\partial x}}_{k_x}+k_y\frac{\partial^2f}{\partial\varphi^2}\underbrace{\frac{\partial\varphi}{\partial y}}_{k_y}+k_z\frac{\partial^2f}{\partial\varphi^2}\cdot\underbrace{\frac{\partial\varphi}{\partial z}}_{k_z}\Big)$$

$$=\cancel{\vec E_0}\left(k_x^2+k_y^2+k_z^2\right)\cancel{\frac{\partial^2f}{\partial\varphi^2}}$$

$$\frac{\partial^2\vec E}{\partial t^2}=\frac{\partial}{\partial t}\Big[\frac{\partial\vec E}{\partial\varphi}\cdot\underbrace{\frac{\partial\varphi}{\partial t}}_{=-\omega}\Big]=-\omega\frac{\partial^2\vec E}{\partial\varphi^2}\cdot\underbrace{\frac{\partial\varphi}{\partial t}}_{-\omega}=\omega^2\cancel{\vec E_0}\cdot\cancel{\frac{\partial^2f}{\partial\varphi^2}}$$

$$\Rightarrow\left(\frac{n}{c}\right)^2\omega^2=k_x^2+k_y^2+k_z^2\equiv|\vec k|^2\equiv k^2$$

or $\underline{\omega=\frac{c}{n}\cdot k}$ ($\propto k$!) → the Ansatz gives us a solution for this relation.

The energy or frequency plotted as a function of the wave vector $\vec k$ is called "dispersion relation".