A1: Stefan-Boltzmann Law

The Stefan-Boltzmann law

$$S = \sigma T^4$$

states that the total energy radiated from a black body per unit surface area and per time is proportional to T^4 with a proportionality constant σ (called the Stefan-Boltzmann constant). This can be derived from Planck's radiation law by integrating the power spectral density $\epsilon(\nu, T)$ over all frequencies and over one half-space,

$$S = \int_0^\infty \mathrm{d}\nu \int_{\text{half-space}} \mathrm{d}\Omega\, \epsilon(\nu, T) \cos\vartheta.$$

(a) Argue why there is a factor cos in the integration. Hint: this is purely geometrical in this case.

(b) Perform the calculation to show that S is proportional to T^4 and determine the Stefan-Boltzmann constant σ. The resulting dimensionless integral can be determined exactly using Mathematica or a table of integrals.

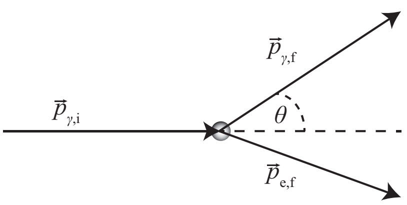
(c) We now use the Stefan-Boltzmann law to estimate the peak electrical power produced by a 1 hectare (=10000 m^2) array of solar panels. For this, one needs the radius of the sun $R_\odot = 6.955 \times 10^8$ m, the average distance between the sun and the earth, $D = 149.6 \times 10^9$ m, and the (surface) temperature of the sun $T_\odot = 5800$ K. The earth's atmosphere reflects and absorbs some of the sun's radiation, so that in average over all wavelengths, about $\eta_{\text{atm}} = 50\%$ of the total power arrives on the earth's surface (on a cloudless day at noon). In addition, solar cells have an imperfect conversion efficiency of optical to electrical power (around $\eta_{\text{con}} = 25\%$) and a finite spectral width (Si-based cells usually work up to a maximum wavelength of 1.2 μm). Thus, only $\eta_{\text{BW}} = 80\%$ of the sun's total radiated power is within the spectral acceptance of these solar cells. Combine all of these facts with the Stefan-Boltzmann law to determine the peak electrical power generated by a hectare array of solar cells.

(d) Use the Stefan-Boltzmann law and the pevious result to estimate the number of detectable photons per second that enters your eye if you would stare directly at the sun (not recommended!). Assume that the iris of the eye is opened to 2 mm diameter in bright light. Since the eye is only sensitive to light in the visible part of the spectrum, only about 40% of the total power entering the eye is detected. Make the simplification that all power is emitted as photons with the peak wavelength $\lambda_{\max} = 500$ nm.

(e) The maximum permissible exposure (MPE) is defined as the maximum intensity (power per area) at the eye that is still considered safe. For visible light, this number is 1 mW/cm^2. Calculate the maximum permissible exposure in units of photons per second entering the eye (with iris diameter equal to 2 mm) for $\lambda = 500$ nm and compare to the answer in the previous problem. Is it dangerous to stare at the sun?

A2: Compton Effect

Consider the experiment by Arthur Compton in 1923 where Compton scattering was observed and explained using a particle (photon) description of light. In Compton scattering, photons of wavelength λ_{i} incident on a material (Compton used graphite) are scattered and leave the material at a wavelength λ_{f} at an angle θ relative to the incident beam, as shown in the figure. The differences in energy and momentum of the incident and scattered photons are absorbed by an electron in the material, which we assume has no initial momentum.



The formula for Compton scattering is

$$\lambda_{\text{f}} - \lambda_{\text{i}} = \frac{h}{m_{\text{e}} c}(1 - \cos\theta),$$

where h is Planck's constant, m_{e} is the electron mass, c is the speed of light. This expression can be derived using energy and momentum conservation and the relativistic expression for the kinetic energy of the electrons. In his experiment, Compton used photons with wavelength 0.0711 nm.

(a) What is the energy of an incident photon?

(b) What angle θ results in the largest shift in the wavelength of the scattered photon relative to the incident photon? What is the wavelength of the scattered photon at this angle?

(c) What is the energy of a photon scattered at this angle?

(d) What are the momenta of the incident and scattered photons?