A1: Wavefunctions

We study the single-particle wavefunction called Loretzian

$$\psi(x) = \frac{N}{(x - x_0)^2 + a^2}$$

with $x_0, a \in \mathbb{R}$.

(a) Find the value of N from the normalization condition.

(b) Calculate the probability for finding the particle in the interval $[x_0 - a, x_0 + a]$.

(c) Guess the average position (including an explanation) $\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 \mathrm{d}x$.

(d) Calculate the variance of the position, $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$.

A2: Ground state of the quantum-mechanical harmonic oscillator

The total energy of a one-dimensional harmonic oscillator is

$$E(x, p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2,$$

with x the position, p the momentum, m the mass of the particle, and $\omega = 2\pi f$ the confinement strength, which corresponds to the classical oscillation frequency.

(a) What is the lowest possible *classical* energy of this oscillator?

(b) The position uncertainty of the harmonic oscillator is $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. The inversion symmetry of the problem implies that $\langle x \rangle = 0$. Find an expression for $\langle x^2 \rangle$ as a function of Δx. Repeat this for the momentum: find $\langle p^2 \rangle$ as a function of Δp.

(c) The expectation value of the energy is

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle.$$

Write down the expectation value of the energy as a function of Δx and Δp.

(d) Heisenberg's uncertainty principle states that $\Delta x \Delta p \geq \frac{1}{2}\hbar$. Find the values of Δx and Δp which simultaneously minimize this uncertainty principle (*i.e.*, $\Delta x \Delta p = \frac{1}{2}\hbar$) and minimize the expectation value of the energy. Express the minimum energy in terms of the classical oscillation frequency ω.

(e) Compare this minimum energy to the minimum energy of a classical harmonic oscillator (problem a). What is the meaning of this result?