Solutions home work problem sheet 5

1a)

$$1 \overset{!}{=} \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} \frac{N^2}{[(x-x_0)^2+a^2]^2} dx = \frac{N^2}{a^4}\int_{-\infty}^{\infty}\frac{1}{[(\frac{x-x_0}{a})^2+1]^2}dx \underset{\substack{\uparrow\\ y=\frac{x-x_0}{a}\\ dy = dx/a}}{=} \frac{N^2}{a^3}\int_{-\infty}^{\infty}\frac{dy}{(y^2+1)^2}$$

$$\underset{\substack{\uparrow\\ \text{table, e.g.}\\ \text{Bronstein p 1053,}\\ \text{formula 58}}}{=} \frac{N^2}{a^3}\left(\frac{\frac{1}{2} y}{y^2+1}+\frac{1}{2}\arctan(y)\right)\Big|_{-\infty}^{\infty} = \frac{N^2}{a^3}\cdot\frac{\pi}{2}$$

$$\Rightarrow \underline{\underline{N=\sqrt{\frac{2a^3}{\pi}}}}$$

b) probability to find the particle between x_0-a and x_0+a:

$$\underline{\underline{P}} = \int_{x_0-a}^{x_0+a}|\Psi(x)|^2dx \underset{\substack{\uparrow\\ \text{see}\\ \text{a)}}}{=} \underbrace{\frac{N^2}{a^3}}_{2/\pi}\left(\frac{\frac{1}{2} y}{y^2+1}+\frac{1}{2}\arctan(y)\right)\Big|_{y=-1}^{y=+1} = \frac{1}{\pi}\left(\left(\frac{1}{2}+\arctan(1)\right)-\left(-\frac{1}{2}+\arctan(-1)\right)\right)$$

$$= \frac{1}{\pi}\left(1+\frac{\pi}{2}\right) = \frac{1}{\pi}+\frac{1}{2} \approx \underline{\underline{82\%}}$$

c)

$$\underline{\underline{\langle x\rangle}} = \int_{-\infty}^{\infty} x\cdot \underbrace{\frac{N^2}{((x-x_0)^2+a^2)^2}}_{\substack{\text{symmetric}\\ \text{around } x_0}} \underset{\substack{\uparrow\\ y=x-x_0}}{=} \int_{-\infty}^{\infty}(y+x_0)\underbrace{\frac{N^2}{(y^2+a^2)^2}}_{\substack{\text{symmetric}\\ \text{around } 0}}dy = \underbrace{\int_{-\infty}^{\infty}\underset{\substack{\uparrow\\ \text{antisym.}}}{y}\cdot \underset{\substack{\uparrow\\ \text{sym.}}}{f_{sym}(y)}\,dy}_{=0} + x_0\cdot\underbrace{\int_{-\infty}^{\infty} f_{sym}\,dy}_{\substack{=1\\ \text{(normalized to 1!)}}}$$

$$= \underline{\underline{x_0}}$$

One can also guess this: set $x_0=0 \Rightarrow \int_{-\infty}^{\infty} x|\Psi(x)|^2dx = 0$, then shift by x_0.

d)

$$var = \langle(x-\langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle - x_0^2$$

$$\underline{\underline{\langle x^2\rangle}} = \int_{-\infty}^{\infty} x^2|\Psi(x)|^2dx \underset{\substack{\uparrow\\ z=x-x_0}}{=} \int_{-\infty}^{\infty}(z+x_0)^2\underbrace{\frac{N^2}{(z^2+a^2)^2}}_{=:f(z)}dz = \int_{-\infty}^{\infty} z^2f(z)dz + \underbrace{\int_{-\infty}^{\infty}\underset{\substack{\uparrow\\ \text{const.}}}{2x_0}\underset{\substack{\uparrow\\ \text{antisym.}}}{z}\overset{\substack{\text{sym.}\\ \downarrow}}{f(z)}dz}_{=0} + x_0^2\underbrace{\int_{-\infty}^{\infty}f(z)dz}_{\substack{=1\\ \text{(normalized } f(z))}}$$

$$= x_0^2 + \overbrace{\frac{2a^3}{\pi}}^{N^2}\int_{-\infty}^{\infty}\frac{z^2}{(z^2+a^2)^2}dz \underset{\substack{\uparrow\\ \text{e.g. Bronstein}\\ \text{p. 1053, formula 66}}}{=} x_0^2+\frac{2a^3}{\pi}\left(-\frac{z}{2(z^2+a^2)}+\frac{1}{2a}\arctan\left(\frac{z}{a}\right)\right)\Big|_{-\infty}^{\infty}$$

$$= x_0^2+\frac{2a^3}{\pi}\cdot\frac{\pi}{2a} = \underline{\underline{x_0^2+a^2}}$$

$$\Rightarrow \underline{\underline{var}} = \langle x^2\rangle - x_0^2 = \underline{\underline{a^2}}$$

2a) $p = 0$ (smallest possible kinetic energy)
$x = 0$ (smallest potential energy)

$$\left.\begin{array}{l} p=0 \\ x=0 \end{array}\right\} \underline{\underline{E_{min}^{(class)} = 0}}$$

b)
$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \overset{\overset{\langle x\rangle = 0 \text{ (same probability for } +x \text{ and } -x)}{\downarrow}}{=} \sqrt{\langle x^2\rangle}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \underset{\underset{\langle p\rangle = 0 \text{ (}+p\text{ and }-p\text{ have same probability for all } p)}{\uparrow}}{=} \sqrt{\langle p^2\rangle}$$

c)
$$\langle E\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\Delta p^2}{2m} + \frac{1}{2}m\omega^2\Delta x^2$$

d) $\Delta x \cdot \Delta p \geq \hbar/2$. For $\Delta x \cdot \Delta p = \hbar/2 \Rightarrow \Delta p = \hbar/(2\Delta x) \Rightarrow \langle E\rangle = \frac{\hbar^2}{2m(2\Delta x)^2} + \frac{1}{2}m\omega^2\Delta x^2$

$$= \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2\Delta x^2$$

minimum energy:
$$\frac{\partial\langle E\rangle}{\partial x} = \frac{\hbar^2}{8m\Delta x^3}\cdot(-2) + m\omega^2\Delta x \overset{!}{=} 0$$

$$m\omega^2 = \frac{\hbar^2}{4m\Delta x^4}$$

$$\Delta x^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$\underline{\underline{\Delta x_{min}}} = \sqrt[4]{\frac{\hbar^2}{4m^2\omega^2}} = \underline{\underline{\sqrt{\frac{\hbar}{2m\omega}}}}$$

$$\Rightarrow \underline{\underline{\Delta p_{min}}} = \hbar/(2\Delta x) = \frac{\hbar}{2\sqrt{\frac{\hbar}{2m\omega}}} = \underline{\underline{\sqrt{\frac{\hbar m\omega}{2}}}}$$

$$\Rightarrow \underline{\underline{\langle E_{min}\rangle}} = \frac{\Delta p_{min}^2}{2m} + \frac{1}{2}m\omega^2\Delta x_{min}^2 = \frac{\hbar\cancel{m}\omega}{2\cancel{m}\cdot 2} + \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\frac{\hbar}{2\cancel{m}\cancel{\omega}} = \underline{\underline{\frac{1}{2}\hbar\omega > 0}}$$

e) This result shows that x and p are conjugated variables, i.e. they are <u>not</u> independent. In contrast to the classical result one <u>cannot</u> minimize the kinetic energy ($\propto p^2$) independently from the potential energy ($\propto x^2$).
The obtained minimum energy is often called "zero-point energy".