A1: Ground state of the hydrogen atom

The total energy (at any given time) of an electron with momentum p at a distance r from a proton (fixed in space) is

$$E(r,p) = \frac{p^2}{2m_\mathrm{e}} - \frac{e^2}{4\pi\epsilon_0 r},$$

with m_e the electron mass, e the unit of charge and ϵ_0 the vacuum permittivity.

(a) Estimate the ground state energy by minimizing the Heisenberg uncertainty relationship. To make "an order of magnitude" estimate, write down the average value of the energy assuming $\langle p^2 \rangle = (\Delta p)^2$ and $\langle \frac{1}{r} \rangle = \frac{1}{\Delta r}$, with $\Delta r \Delta p \geq \hbar$. Then find the ground state values Δr_min, Δp_min, and E_min (see also problem sheet 5 for ideas).

(b) Calculate the speed of an electron, $v_\mathrm{min} = \Delta p_\mathrm{min}/m_\mathrm{e}$, and compare it to the speed of light. Is a non-relativistic theory of the hydrogen atom justified?

(c) Compare the above results with the old-fashioned atomic model with circular orbits: assuming that the electron moves on a circular orbit of radius Δr_min with velocity v_min, what will be its orbital frequency? Compare this orbital frequency to the frequency $|E_\mathrm{min}|/h$.

A2: Time–Frequency uncertainty relation

Given a time-dependent signal

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ \sin(\Omega t) e^{-t/\tau} & \text{for } t \geq 0. \end{cases}$$

(a) Plot this function for $\Omega\tau = 30$. What is the ring-down time of this oscillation, i.e., in what time does the amplitude of the fast oscillation decay to $1/e$ of the original value?

(b) Calculate the Fourier transform $\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\mathrm{i}\omega t} \mathrm{d}t$.

(c) Plot the power spectrum $|\tilde{f}(\omega)|^2$.

(d) Calculate the spectral pulse width $\Delta\omega$, defined as the "full width at half maximum" (FWHM) of the peak(s) in the power spectrum.

(e) Calculate $\Delta\omega\Delta t$ and interpret the result in the context of uncertainty relations.