Solutions 6

1a) The average energy reads

$$\langle E\rangle = \left\langle \frac{p^2}{2m}\right\rangle - \left\langle \frac{e^2}{4\pi\varepsilon_0 r}\right\rangle = \frac{1}{2m}\langle p^2\rangle - \frac{e^2}{4\pi\varepsilon_0}\left\langle\frac{1}{r}\right\rangle \overset{\text{assumption in text}}{=} \frac{1}{2m}\Delta p^2 - \frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{\Delta r}.$$

From the uncertainty relation we find $\Delta p = \hbar/\Delta r$

$$\Rightarrow \langle E\rangle = \frac{\hbar^2}{2m}\frac{1}{\Delta r^2} - \frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{\Delta r}.$$

We obtain the ground state parameters by minimizing $\langle E\rangle$ with respect to Δr:

$$\frac{\partial\langle E\rangle}{\partial \Delta r} \overset{!}{=} 0 = -2\frac{\hbar^2}{2m}\cdot\frac{1}{\Delta r_m^3} + \frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{\Delta r_m^2} \quad \Big|\cdot \Delta r_m^2, \ldots$$

$$\underline{\underline{\Delta r_{min}}} = \frac{4\pi\varepsilon_0}{e^2}\cdot\frac{\hbar^2}{m} \approx \underline{\underline{0.529\,\text{Å}}}$$

$$\Rightarrow \underline{\underline{\Delta p_{min}}} = \hbar/\Delta r_{min} = \frac{me^2}{4\pi\varepsilon_0\hbar} \approx \underline{\underline{1.99\cdot 10^{-24}\,\text{mkg/s}}}$$

$$\Rightarrow \underline{\underline{E_{min}}} = \frac{\Delta p_m^2}{2m} - \frac{e^2}{4\pi\varepsilon_0\Delta r_m} = \frac{m^2e^4}{2m\cdot(4\pi)^2\varepsilon_0^2\hbar^2} - \frac{e^2\cdot e^2\cdot m^2}{4\pi\varepsilon_0\, 4\pi\varepsilon_0^2\cdot\hbar^2} = -\frac{me^4}{2(4\pi)^2\varepsilon_0^2\hbar^2} \approx -2.18\cdot 10^{-18}\,\text{J} = \underline{\underline{-13.6\,\text{eV}}}$$

b) $$v_{min} = \frac{\Delta p_{min}}{m} = \frac{e^2}{4\pi\varepsilon_0\hbar} \approx 2.19\cdot 10^6\,\text{m/s} =: \alpha\cdot \underset{\uparrow\ \text{speed of light}}{c}\ ; \quad \alpha\approx\frac{1}{137}: \text{fine structure constant}$$

$\Rightarrow v_{min} \ll c \rightarrow$ relativistic effects can be neglected in this example.

c) circular orbits of radius Δr_{min}

$$\Rightarrow \text{orbital frequency } \underline{\underline{f_{co}}} = \left(\frac{2\pi\cdot\Delta r_{min}}{v_{min}}\right)^{-1} = \frac{\frac{e^2}{4\pi\varepsilon_0\hbar}}{2\pi\cdot\frac{4\pi\varepsilon_0}{e^2}\cdot\frac{\hbar^2}{m}} = \frac{me^4}{32\pi^3\varepsilon_0^2\hbar^3} \approx \underline{\underline{6.58\cdot 10^{15}\,\text{Hz}}}$$

From above: $$f = \frac{E_{min}}{2\pi\hbar} = \frac{me^4}{64\pi^3\varepsilon_0^2\hbar^3} = \tfrac{1}{2}f_{co}$$

$\Rightarrow$ the concept of a point particle orbiting the core on a classical trajectory is not applicable in the "quantum world".

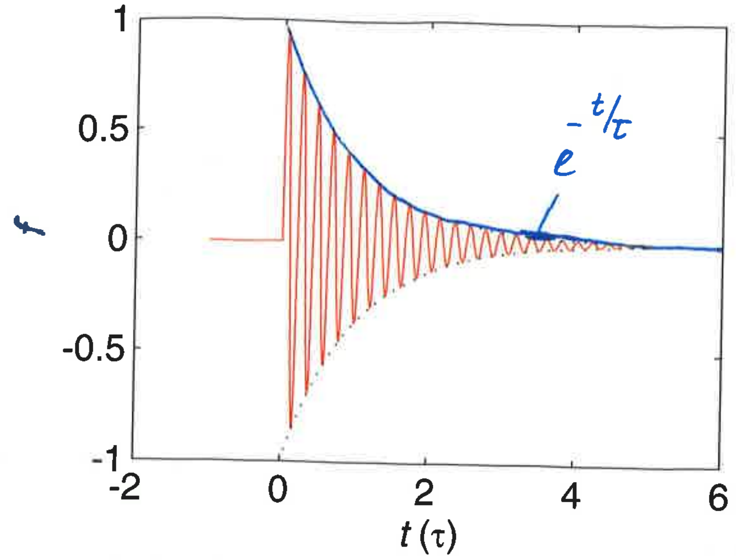
②

2a) $f(t) = \begin{cases} 0 & \text{for } t<0 \\ \sin(\Omega t)e^{-t/\tau} & \text{for } t \geq 0 \end{cases}$

The ring-down time is τ.

In Matlab:

```
τ=1; Ω=30;
t=-1:0.01:6;
f=sin(Ω*t).*exp(-t/τ);
ind=find(t<0);
f(ind)=0;
plot(t,f,'r');
```



b) $\underline{\underline{\tilde{f}(\omega)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sin(\Omega t) e^{-t/\tau} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{2i}\left[e^{i\Omega t} - e^{-i\Omega t}\right] e^{-(1/\tau + i\omega)t} dt$$

$$= \frac{1}{2i\sqrt{2\pi}} \int_0^{\infty} e^{(i\Omega - 1/\tau - i\omega)t} dt - \frac{1}{2i\sqrt{2\pi}} \int_0^{\infty} e^{(-i\Omega - 1/\tau - i\omega)t} dt$$

$$= \frac{1}{2i\sqrt{2\pi}} \left.\frac{e^{i\Omega - 1/\tau - i\omega}}{i\Omega - 1/\tau - i\omega}\right|_0^{\infty} - \frac{1}{2i\sqrt{2\pi}} \left.\frac{e^{-i\Omega - 1/\tau - i\omega}}{-i\Omega - 1/\tau - i\omega}\right|_0^{\infty} = \frac{1}{2i\sqrt{2\pi}}\left[0 - \frac{1}{i\Omega - 1/\tau - i\omega} - 0 + \frac{1}{-i\Omega - 1/\tau - i\omega}\right]$$

$$= \frac{1}{2i\sqrt{2\pi}} \frac{i\Omega + \cancel{1/\tau} + \cancel{i\omega} + i\Omega - \cancel{1/\tau} - \cancel{i\omega}}{(i\Omega - 1/\tau - i\omega)(-i\Omega - 1/\tau - i\omega)} = \frac{1}{i\sqrt{2\pi}} \frac{\Omega}{[(1/\tau + i\omega) - i\Omega][(1/\tau + i\omega) + i\Omega]} = \underline{\underline{\frac{1}{i\sqrt{2\pi}} \frac{\Omega}{(1/\tau + i\omega)^2 + \Omega^2}}}$$

2c) $\underline{|\tilde{f}(\omega)|^2} = \tilde{f}^* f = \frac{1}{2\pi} \frac{\Omega^2}{[(1/\tau - i\omega)^2 + \Omega^2][(1/\tau + i\omega)^2 + \Omega^2]} = \frac{1}{2\pi} \frac{\Omega^2}{[1/\tau^2 - \frac{2i\omega}{\tau} - \omega^2 + \Omega^2][1/\tau^2 + \frac{2i\omega}{\tau} - \omega^2 + \Omega^2]}$

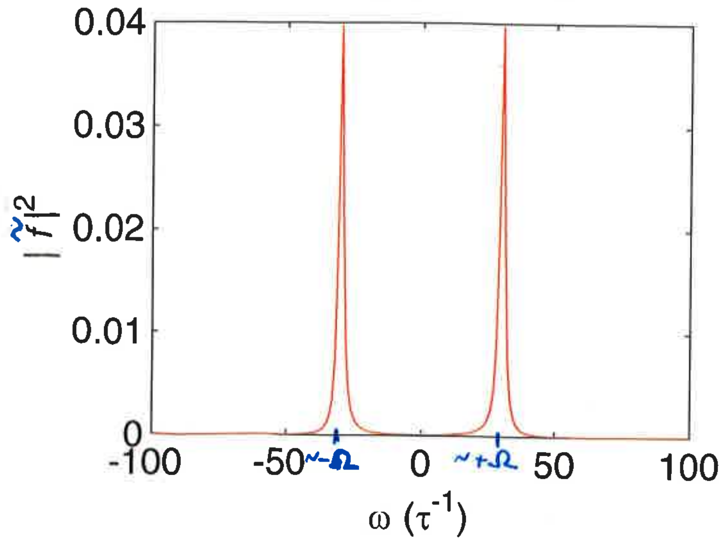
$$= \frac{1}{2\pi} \frac{\Omega^2\tau^4}{[1 - 2i\omega\tau - \omega^2\tau^2 + \Omega^2\tau^2][1 + 2i\omega\tau - \omega^2\tau^2 + \Omega^2\tau^2]} = \frac{1}{2\pi} \frac{\Omega^2\tau^4}{\ldots}$$

$1 - \cancel{2i\omega\tau} - \omega^2\tau^2 + \Omega^2\tau^2$
$+ \cancel{2i\omega\tau} + 4\omega^2\tau^2 - \cancel{2i\omega^3\tau^3} + \cancel{2i\Omega^2\omega\tau^3}$
$+ \cancel{2i\omega^3\tau^3} - \omega^2\tau^2 + \omega^4\tau^4 - \Omega^2\omega^2\tau^4$
$+ \Omega^2\tau^2 - \cancel{2i\omega\tau^3\Omega^2} - \Omega^2\omega^2\tau^4 + \Omega^4\tau^4$

$$= \frac{1}{2\pi} \frac{\Omega^2\tau^4}{1 - 2(\omega\tau)^2 + 2(\Omega\tau)^2 + (\omega\tau)^4 - 2\omega^2\Omega^2\tau^4 + (\Omega\tau)^4 + 4\omega^2\tau^2}$$

$$= \underline{\underline{\frac{1}{2\pi} \frac{\Omega^2\tau^4}{1 + 2\tau^2(\omega^2 + \Omega^2) + \tau^4(\omega^2 - \Omega^2)^2}}}$$

plot in Matlab:
similar as in 2a



2d) step 1: find positions of maxima in $|\tilde{f}|^2$ found in 2c.

The maximum corresponds to the minimum in the denominator

$$\Rightarrow \frac{\partial}{\partial\omega}\left[1+2\tau^2(\omega^2+\Omega^2)+\tau^4(\omega^2-\Omega^2)^2\right] = \cancel{4}\tau^2\omega + \cancel{4}\tau^4\cancel{\omega}(\omega^2-\Omega^2) \overset{!}{=} 0$$

$$\Rightarrow \quad 1+\tau^2(\omega_m^2-\Omega^2) \overset{!}{=} 0 \qquad \text{or} \qquad \omega_m^2 = -\frac{1}{\tau^2}+\Omega^2 = \frac{(\Omega\tau)^2-1}{\tau^2}$$

$$\underline{\underline{\omega_m = \pm\frac{\sqrt{(\Omega\tau)^2-1}}{\tau}}}$$

step 2: maximum value:

$$\underline{\underline{|\tilde{f}(\omega_m)|^2}} = \frac{1}{2\pi}\frac{\Omega^2\tau^4}{1+2\tau^2\left[\frac{\Omega^2\tau^2-1}{\tau^2}+\Omega^2\right]+\tau^4\left[\frac{\Omega^2\tau^2-1}{\tau^2}-\Omega^2\right]^2}$$

$$= \frac{1}{2\pi}\frac{\Omega^2\tau^4}{1+2\Omega^2\tau^2-2+2\Omega^2\tau^2+(\Omega^2\tau^2-1)^2-2\tau^2\Omega^2[\Omega^2\tau^2-1]+\Omega^4\tau^4}$$

$$= \frac{1}{2\pi}\frac{\Omega^2\tau^4}{\cancel{1}+2\Omega^2\tau^2-\cancel{2}+2\Omega^2\tau^2+\Omega^4\tau^4-2\Omega^2\tau^2+\cancel{1}-2\tau^4\Omega^4+2\tau^2\Omega^2+\Omega^4\tau^4}$$

$$= \frac{1}{2\pi}\frac{\Omega^2\tau^4}{4\Omega^2\tau^2} = \underline{\underline{\frac{\tau^2}{8\pi}}}$$

step 3: find frequency ω_h with $|\tilde{f}(\omega_h)|^2 = \frac{1}{2}|\tilde{f}(\omega_m)|^2 = \frac{\tau^2}{16\pi}$

$$\Rightarrow \frac{\tau^2}{16\pi} \overset{!}{=} \frac{1}{2\pi}\frac{\Omega^2\tau^4}{1+2\tau^2(\omega_h^2+\Omega^2)+\tau^4(\omega_h^2-\Omega^2)^2}$$

$$\Rightarrow 8\Omega^2\tau^2 \overset{!}{=} 1+2\tau^2(\omega_h^2+\Omega^2)+\omega_h^4\tau^4-2\tau^4\omega_h^2\Omega^2+\Omega^4\tau^4 = 1+2\omega_h^2\tau^2+2\Omega^2\tau^2-\omega_h^4\tau^4-2\Omega^2\tau^4\omega_h^2+\Omega^4\tau^4 \quad \Big|:\tau^4$$

$$\text{or} \quad 0 = \frac{1}{\tau^4}+\frac{2\Omega^2}{\tau^2}+\Omega^4-\frac{8\Omega^2}{\tau^2}+\frac{2\omega_h^2}{\tau^2}-2\Omega^2\omega_h^2-\omega_h^4 = \omega_h^4+2\omega_h^2\left[1/\tau^2-\Omega^2\right]+\left[1/\tau^4-\frac{6\Omega^2}{\tau^2}+\Omega^4\right]$$

$$\underset{\text{complete the square}}{=} \left(\omega_h^2+\left[1/\tau^2-\Omega^2\right]\right)^2-\left[1/\tau^2-\Omega^2\right]^2+1/\tau^4-\frac{6\Omega^2}{\tau^2}+\Omega^4$$

$$= \left(\omega^2+\left[1/\tau^2-\Omega^2\right]\right)^2-\cancel{1/\tau^4}+\frac{2\Omega^2}{\tau^2}-\cancel{\Omega^4}+\cancel{1/\tau^4}-\frac{6\Omega^2}{\tau^2}+\cancel{\Omega^4}$$

$$= \left(\omega^2+\left[1/\tau^2-\Omega^2\right]\right)^2-\frac{4\Omega^2}{\tau^2}$$

$$\Rightarrow \quad \omega_h^2+(1/\tau^2-\Omega^2) = \pm\frac{2\Omega}{\tau}$$

$$\omega_h^2 = \Omega^2\pm\frac{2\Omega}{\tau}-\frac{1}{\tau^2} = \frac{1}{\tau^2}\,\Omega^2\tau^2\pm2\Omega\tau-1$$

$$\underline{\underline{\omega_h}} = \pm\frac{1}{\tau}\sqrt{\Omega^2\tau^2\pm2\Omega\tau-1}$$

step 4 Full width half maximum (FWHM):

$\sqrt{1\pm x}\approx 1\pm\frac{1}{2}x$ for small $x\ll1$

$$\underline{\underline{\Delta\omega}} = \frac{1}{\tau}\sqrt{\Omega^2\tau^2\underset{\uparrow}{+}2\Omega\tau-1}-\frac{1}{\tau}\sqrt{\Omega^2\tau^2\underset{\uparrow}{-}2\Omega\tau-1} \underset{\Omega\tau\gg1}{\approx} \frac{\Omega\tau}{\tau}\sqrt{1+\frac{2}{\Omega\tau}}-\frac{\Omega\tau}{\tau}\sqrt{1-\frac{2}{\Omega\tau}} \approx \Omega\left(1+\frac{1}{\Omega\tau}-1+\frac{1}{\Omega\tau}\right) = \underline{\underline{\frac{2}{\tau}}}$$

2e) With $\Delta t=\tau$: $\underline{\underline{\Delta\omega\cdot\Delta t\cong2}}$, i.e. the wider in real time, the sharper in the Fourier space, similar as for uncertainty rel. (Gaussians).