A1: Infinite Square Well - time dependent states

In the lecture (and soon the tutorial class) you solved the eigenvalue problem for a particle in an infinitely deep square well potential. The general solution to the time-independent Schrödinger equation is a superposition of the energy eigenstates $\phi_n(x)$.

At time $t = 0$, the particle is in an arbitrary state

$$\psi(x, t = 0) = \sum_n c_n \phi_n(x)$$

determined by the (constant) coefficients c_n.

(a) Write down an expression for the time evolution $\psi(x, t)$ for the superposition state $\psi(x, t = 0)$.

(b) Now we focus on two specific initial superposition states of the particle: (i) the particle in the ground state

$$\psi_{\mathrm{i}}(x, t = 0) = \phi_1(x)$$

and (ii) the particle in a superposition of the ground and first excited state

$$\psi_{\mathrm{ii}}(x, t = 0) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + \phi_2(x)\right).$$

Determine $\psi(x, t)$ and $|\psi(x, t)|^2$ for both superposition states.

(c) For both states determine the expectation value of the particle's position as a function of time

$$\langle x(t)\rangle = \int dx\, \psi^*(x, t)\, x\, \psi(x, t).$$

(d) Why is $\langle x(t)\rangle$ time-dependent in one case and not the other?

A2: Quantum Continuity Equation

It is intuitively clear that the total probability of finding a given particle anywhere is conserved (as long as the particle does not vanish in a chemical reaction or in a collision experiment). This means that for any given time t

$$\int dV\, \psi^*(\vec{r}, t)\, \psi(\vec{r}, t) = 1$$

holds (the integral is taken over the full real space). Like other conserved quantities (e.g. charge or mass), the transport of probability in space and time is governed by a continuity equation.

(a) Derive the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0,$$

for $\rho = |\psi(\vec{r}, t)|^2$, using Schrödinger's equation. Give an explicit expression for the probability current density $\vec{j}$.

(b) Express in words what the continuity equation describes.