Solutions to Homework 7

1a) $\Psi(x,t=0) = \sum_n c_n \cdot \phi_n(x)$

For the eigenstates the time-dependent Schrödinger equation $i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$ reads: $i\hbar\frac{\partial}{\partial t}\phi_n(x,t) = H\phi_n \underset{\uparrow}{=} E_n\phi_n$. Direct integration gives $\phi_n(x,t) = \underbrace{e^{-\frac{iE_n\Delta t}{\hbar}}}_{\text{just a phase factor!}} \cdot \phi_n(x, t+\Delta t)$.

eigenstate!

$$\Rightarrow \Psi(x,t) = \sum_n c_n \cdot \phi_n(x,t) = \sum_n c_n \cdot \phi_n(x,t=0) \cdot e^{-\frac{iE_n t}{\hbar}} \quad \left[\text{with } E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \text{ from above}\right]$$

b) (i) $\Psi(x,t=0) = \phi_{n=1}(x)$ (ground state) $\Leftrightarrow c_1 = 1,\ c_i = 0\ \forall i \neq 1$

$$\Rightarrow \Psi(x,t) = e^{-\frac{iE_1 t}{\hbar}} \cdot \phi_1(x) = \sqrt{\frac{2}{L}}\, e^{-\frac{i\pi^2\hbar^{\cancel{2}}}{2m\cancel{\hbar}L^2}\cdot t} \cdot \sin\left(\frac{\pi x}{L}\right)$$

$$|\Psi(x,t)|^2 = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right) \quad (|e^{ia}|^2 = 1 \text{ for } a \in \mathbb{R})$$

(ii) $\Psi(x,t=0) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + \phi_2(x)\right]$

$$\Rightarrow \Psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-\frac{iE_1 t}{\hbar}} \cdot \phi_1(x) + e^{-\frac{iE_2 t}{\hbar}}\phi_2(x)\right] = \frac{1}{\sqrt{2}} e^{-\frac{iE_1 t}{\hbar}}\left[\phi_1(x) + e^{-\frac{i\overbrace{(E_2 - E_1)}^{\Delta E}t}{\hbar}}\phi_2(x)\right]$$

$$\underset{\uparrow}{=} \frac{1}{\sqrt{L}} e^{-\frac{i\pi^2\hbar^{\cancel{2}}}{2mL^2\cancel{\hbar}}\cdot t}\left[\sin\left(\frac{\pi x}{L}\right) + e^{-\frac{i3\pi^2\hbar}{2mL^2}} \cdot \sin\left(\frac{2\pi x}{L}\right)\right] \quad \leftarrow n=2$$

$\Delta E = E_2 - E_1 = \frac{\pi^2\hbar^2(2^2-1^2)}{2mL^2}$

$$|\Psi(x,t)|^2 \underset{\uparrow}{=} \frac{1}{2}\left(\phi_1^* + e^{\frac{i\Delta E t}{\hbar}}\phi_2^*\right)\cdot\left(\phi_1 + e^{-\frac{i\Delta E t}{\hbar}}\phi_2\right) = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + \phi_1\phi_2^* e^{i\Delta E t/\hbar} + \phi_1^*\phi_2 e^{-i\Delta E t/\hbar}\right)$$

global phase $\to 1$

$$\underset{\uparrow}{=} \frac{1}{2}\left[\phi_1^2 + \phi_2^2 + \phi_1\phi_2\left(e^{i\Delta E t/\hbar} + e^{-i\Delta E t/\hbar}\right)\right] = \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_2^2 + \phi_1\phi_2 \cdot \cos\left(\Delta E/\hbar \cdot t\right)$$

$\phi_1, \phi_2 \in \mathbb{R}$

$$= \frac{1}{L}\left[\underbrace{\sin^2\left(\frac{\pi x}{L}\right)}_{\phi_1^2} + \underbrace{\sin^2\left(\frac{2\pi x}{L}\right)}_{\phi_2^2} + 2\cdot\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cdot\cos\left(\frac{3\pi^2\hbar}{2mL^2}\cdot t\right)\right]$$

1c) $\langle x(t)\rangle = \int \Psi^*(x,t)\cdot x\cdot \Psi(x,t)\,dx$

(i) $\underline{\langle x_i(t)\rangle} = \frac{2}{L}\int_0^L x\cdot\sin^2\left(\frac{\pi x}{L}\right)dx \underset{\substack{y = x - L/2 \\ dy/dx = 1}}{=} \frac{2}{L}\cdot\int_{-L/2}^{L/2}(y+L/2)\cdot\sin^2\Big[\underbrace{\tfrac{\pi}{L}(y - L/2)}_{\frac{\pi y}{L}-\frac{\pi}{2}}\Big]dy = \underbrace{\frac{2}{L}\int_{-L/2}^{L/2} y\cdot\cos^2\left(\frac{\pi y}{L}\right)dy}_{=0} + \frac{2}{L}\cdot\frac{L}{2}\int_{-L/2}^{L/2}\cos^2\left(\frac{\pi y}{L}\right)dy$

asym. symmetric

$= \left[\frac{1}{2}y + \frac{L}{4\pi}\cdot\sin\left(\frac{2\pi y}{L}\right)\right]_{-L/2}^{L/2} = \underline{\underline{\frac{L}{2}}}$ ~~(or directly Bronstein)~~

Bronstein p. 1069 eq. 314

One could have guessed also this result: the time evolution is only a phase factor!

(ii) $\underline{\underline{\langle x_{ii}(t)\rangle}} = \int_0^L x\cdot|\Psi_{ii}(x,t)|^2dx = \frac{1}{L}\int_0^L x\cdot\sin^2\left(\frac{\pi x}{L}\right)dx + \frac{1}{L}\int_0^L x\cdot\sin^2\left(\frac{2\pi x}{L}\right)dx + \frac{2}{L}\int_0^L x\cdot\sin\left(\frac{\pi x}{L}\right)\cdot\sin\left(\frac{2\pi x}{L}\right)\cdot\overbrace{\cos\left(\frac{3\pi^2\hbar}{2mL^2}t\right)}^{\text{indep. of } x}dx$

$= \frac{1}{2L}\int_0^L x\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]dx + \frac{1}{2L}\cdot\int_0^L x\left[1-\cos\left(\frac{4\pi x}{L}\right)\right]dx + \frac{2}{L}\cdot\cos\left(\frac{3\pi^2\hbar}{2mL^2}t\right)\cdot\int_0^L x\sin\left(\frac{\pi x}{L}\right)\cdot\underbrace{2\cdot\sin\left(\frac{\pi x}{L}\right)\cdot\cos\left(\frac{\pi x}{L}\right)}_{\sin\left(\frac{2\pi x}{L}\right)}dx$

$\sin\left(\frac{2\pi x}{L}\right)$: Bronstein p. 81, eq. 2.90

see above

$= \frac{L}{4} + \frac{L}{4} + \frac{4}{L}\cos\left(\frac{3\pi^2\hbar}{2mL^2}t\right)\cdot\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)\cdot\cos\left(\frac{\pi x}{L}\right)dx$ $\left\{\begin{array}{l} u = x \\ v' = \sin^2\cos \leftrightarrow v = \frac{1}{3}\frac{L}{\pi}\cdot\sin^3\left(\frac{\pi x}{L}\right)\end{array}\right\}$

partial integration

$= \frac{L}{2} + \frac{4}{L}\cos\left(\frac{3\pi^2\hbar t}{2mL^2}\right)\cdot\left\{uv - \int u'v\,dx\right\}$

$= \frac{L}{2} + \frac{4}{L}\cos\left(\frac{3\pi^2\hbar t}{2mL^2}\right)\Big\{\underbrace{x\cdot\frac{L}{3\pi}\sin^3\left(\frac{\pi x}{L}\right)\Big|_0^L}_{=0} - \int_0^L\frac{L}{3\pi}\sin^3\left(\frac{\pi x}{L}\right)dx\Big\} = \frac{L}{2} + \frac{4}{L}\cos\left(\frac{3\pi^2\hbar t}{2mL^2}\right)\Big\{\underbrace{\frac{L^2}{3\pi^2}\cos\left(\frac{\pi x}{L}\right)\Big|_0^L}_{-2} - \underbrace{\frac{L^2}{9\pi^2}\cos^3\left(\frac{\pi x}{L}\right)\Big|_0^L}_{-2}\Big\}$

Bronstein p. 1067, eq. 276

$= \frac{L}{2} + \frac{4}{L}\cos\left(\frac{3\pi^2\hbar t}{2mL^2}\right)\cdot\left\{-\frac{2L^2}{3\pi^2} + \frac{2L^2}{9\pi^2}\right\}$

$= \underline{\frac{L}{2} - \frac{16L}{9\pi^2}\cdot\cos\left(\frac{3\pi^2\hbar}{2mL^2}\cdot t\right)}$

d) Case (i) is an eigenstate with a "trivial" time dependence (global phase factor).
In contrast, the phase factors of the two superposed states in case (ii) vary with different frequencies $\omega_i = \frac{E_i}{\hbar}$), which leads to a continuous ($\propto t$) change of the <u>relative</u> phase between the two states at the frequency $\frac{\Delta E}{\hbar}$.
The sum of the two states then has intervalls where they add up to a larger value and intervalls where they cancel. With the relative phase these intervalls change, which leads to the oscillation in the particle position.

2a) We aim to get an expression for $\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}(\psi^*\psi)$

real: $U \in \mathbb{R}$

Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + U\psi \Rightarrow -i\hbar\underbrace{\frac{\partial}{\partial t}}_{t\in\mathbb{R}}\psi^* = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi^* + U\psi^*$

$\in\mathbb{R} \rightarrow$ can be swapped (does not apply for operators!)

$$\Rightarrow \underline{\underline{\frac{\partial}{\partial t}[\psi^*\psi]}} = \psi^*\frac{\partial}{\partial t}\psi + \left(\frac{\partial}{\partial t}\psi^*\right)\psi = \frac{1}{i\hbar}\psi^*\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + U\psi\right] + \frac{1}{-i\hbar}\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi^* + U\psi^*\right]\psi$$

$$= \frac{\hbar i}{2m}\left[\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right] + \frac{1}{i\hbar}\underbrace{\left[\psi^* U\psi - \psi^* U\psi\right]}_{\substack{U\in\mathbb{R}\text{ (number)}\\ \rightarrow = 0}} = \underline{\underline{\frac{\hbar i}{2m}\left[\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right]}}$$

From the product rule of differentiation we obtain

$\vec{\nabla}[A\cdot\vec{\nabla}B] = \vec{\nabla}A\cdot\vec{\nabla}B + A\cdot\vec{\nabla}^2B$ or $A\vec{\nabla}^2B = \vec{\nabla}[A\cdot\vec{\nabla}B] - \vec{\nabla}A\cdot\vec{\nabla}B$, so that

$\psi^*\vec{\nabla}^2\psi = \vec{\nabla}[\psi^*\vec{\nabla}\psi] - \underline{\vec{\nabla}\psi^*\vec{\nabla}\psi}$ and $\psi\vec{\nabla}^2\psi^* = \vec{\nabla}[\psi\cdot\vec{\nabla}\psi^*] - \underline{\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*}$

and $\underline{\frac{\partial}{\partial t}[\psi^*\psi]} = \frac{\hbar i}{2m}\left[\vec{\nabla}[\psi^*\vec{\nabla}\psi] - \vec{\nabla}[\psi\vec{\nabla}\psi^*]\right] = \underline{\underline{\frac{\hbar i}{2m}\vec{\nabla}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]}}$

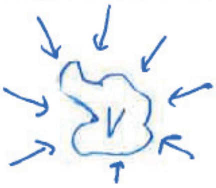
$$\Rightarrow \frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}[\psi^*\psi] \Rightarrow \frac{\partial\rho}{\partial t} - \frac{\hbar i}{2m}\vec{\nabla}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right] = \frac{\partial\rho}{\partial t} + \frac{\hbar}{2mi}\vec{\nabla}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right] \overset{!}{=} 0$$

This can be rewritten with the current density for probability

$\underline{\underline{\vec{j} := \frac{\hbar}{2mi}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]}}$ to obtain the continuity equation $\underline{\underline{\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} = 0}}$

b)

Interpretation: If particles flow into a (small) region of space, the probability of finding a particle in this volume increases



flowing in $\Leftrightarrow \vec{\nabla}\vec{j} < 0 \Rightarrow \frac{\partial\rho}{\partial t} > 0$ (similarly for flowing out)