Beweis der Unschärferelation

Skalarprod. $\Psi, \varphi \mapsto \langle \Psi | \varphi \rangle$
(anderes Symbol)

$$\Delta A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$
$$= \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{A} - \langle \hat{A} \rangle) \Psi \rangle \quad \text{da hermitisch}$$
$$= \langle f | f \rangle \quad \text{mit } f := (\hat{A} - \langle \hat{A} \rangle) \Psi$$

$$\Delta B^2 = \langle g | g \rangle \quad \text{mit } g := (\hat{B} - \langle \hat{B} \rangle) \Psi$$

Schwarzsche Ungl.: $\Delta A^2 \cdot \Delta B^2 = \underline{\langle f | f \rangle \cdot \langle g | g \rangle \geq |\langle f | g \rangle|^2}$ (*)

für bel. $z \in \mathbb{C}$ gilt:

$$|z|^2 = |Re(z)|^2 + |Im(z)|^2 \geq |Im(z)|^2 = \left| \frac{z - z^*}{2i} \right|^2 = \frac{1}{4} |z - z^*|^2$$

mit $z := \langle f | g \rangle$ und von (*) weiter:

$$\underline{\underline{\Delta A^2 \cdot \Delta B^2}} \geq |\langle f | g \rangle|^2 = |z^2| \geq \underline{\frac{1}{4} |\langle f | g \rangle - \langle g | f \rangle|^2}$$

$$\langle f | g \rangle = \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$
$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$
$$= \langle \Psi | (\hat{A}\hat{B} - \hat{A} \langle \hat{B} \rangle - \langle \hat{A} \rangle \hat{B} + \langle \hat{A} \rangle \langle \hat{B} \rangle) \Psi \rangle$$
$$= \langle \hat{A}\hat{B} \rangle - \langle \hat{B} \rangle \langle \hat{A} \rangle - \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle} + \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle}$$

ebenso:

$$\langle g | f \rangle = \langle \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \quad \Rightarrow$$

$$\Delta A^2 \cdot \Delta B^2 \geq \frac{1}{4} |\langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle|^2 = \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2$$

$\Longrightarrow$

$$\underline{\underline{\Delta A \cdot \Delta B \geq \frac{1}{2} \langle [A, B] \rangle}} \quad \blacksquare$$