A1: Angular Momentum

The angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in Cartesian coordinates are

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

(a) Prove the following commutation relations for these operators by evaluating explicitly the commutators on a wavefunction $\psi(\mathbf{r})$. The commutation relations can be written as

$$[\hat{L}_k, \hat{L}_l] = i\hbar\epsilon_{klm}\hat{L}_m$$

with ϵ_{klm} the Levi-Civita symbol, which is +1 for even permutations of (k, l, m), [e.g. (x, y, z) or (z, x, y)] and −1 for odd permutations [e.g. (x, z, y) or (y, x, z)]. It is 0 if any index is repeated [e.g. (x, y, y)]. Please note that this symbol just allows for a very short notation and does not change any of the physics!

A2: Hermitian Operators

(a) Show that a linear combination of two eigenstates of an observable is also an eigenstate if both have the same eigenvalue (these states are said to be degenerate), but not if they have different eigenvalues.

(b) Show that if $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$ for all functions h in the corresponding Hilbert space, then $\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle$ for all f and g. Hint: evaluate the expression for $h = f + g$ and for $h = f + ig$.