Physics III   <u>Series 9</u>   1

1a) $\vec{L} = \vec{r} \wedge \vec{p} = -i\hbar\,(\vec{r} \wedge \vec{\nabla}) = -i\hbar \begin{pmatrix} x \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = -i\hbar \begin{pmatrix} y\partial_z - z\partial_y \\ z\partial_x - x\partial_z \\ x\partial_y - y\partial_x \end{pmatrix} =: \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ (repetition)

The easiest way is to calculate each commutation relation explicitly. A more formal treatment can be found below.

$$[L_x, L_y]\psi = L_xL_y\psi - L_yL_x\psi = -\hbar^2(y\partial_z - z\partial_y)(z\partial_x - x\partial_z)\psi + \hbar^2(z\partial_x - x\partial_z)(y\partial_z - z\partial_y)\psi$$

$$= -\hbar^2[y\cdot\partial_z(z\partial_x) - y\partial_z(x\partial_z) - z\partial_y(z\partial_x) + z\partial_y(x\partial_z) - z\partial_x(y\partial_z) + z\partial_x(z\partial_y) + x\partial_z(y\partial_z) - x\partial_z(z\partial_y)]\psi$$

$$= -\hbar^2[\overbrace{y\cdot\partial_x + \cancel{yz\partial_z\partial_x}} - \cancel{yx\partial_z^2} - z^2\partial_y\partial_x + zx\partial_y\partial_z - \cancel{zy\partial_x\partial_z} + z^2\partial_x\partial_y + \cancel{xy\partial_z^2} - \overbrace{x\partial_y - xz\partial_z\partial_y}]\psi$$

∂_i's commute, x,y,z alone commute!

$$\hat{=} -\hbar^2[y\partial_x - x\partial_y]\psi = -i\hbar[-i\hbar\{y\partial_x - x\partial_y\}]\psi \equiv +i\hbar L_z\psi$$

$$\Rightarrow \underline{\underline{[L_x, L_y] = +i\hbar L_z}}$$

similarly:

$$[L_y, L_z]\psi = L_yL_z\psi - L_zL_y\psi = -\hbar^2(z\partial_x - x\partial_z)(x\partial_y - y\partial_x)\psi + \hbar^2(x\partial_y - y\partial_x)(z\partial_x - x\partial_z)\psi$$

$$= -\hbar^2[z\partial_x(x\partial_y) - z\partial_x(y\partial_x) - x\partial_z(x\partial_y) + x\partial_z(y\partial_x) - x\partial_y(z\partial_x) + x\partial_y(x\partial_z) + y\partial_x(z\partial_x) - y\partial_x(x\partial_z)]\psi$$

$$= -\hbar^2[\overbrace{z\partial_y + \cancel{zx\partial_x\partial_y}} - \cancel{zy\partial_x^2} - x^2\partial_z\partial_y + xy\partial_z\partial_x - \cancel{xz\partial_y\partial_x} + x^2\partial_y\partial_z + \cancel{zy\partial_x^2} - \overbrace{y\partial_z - yx\partial_x\partial_z}]\psi$$

$$= -\hbar^2[z\partial_y - y\partial_z]\psi = -i\hbar[-i\hbar(z\partial_y - y\partial_z)]\psi = +i\hbar L_x\psi$$

$$\Rightarrow \underline{\underline{[L_y, L_z] = +i\hbar L_x}}$$

other elements: $[L_x, L_x] = L_x^2 - L_x^2 = 0$ (similar for other components)

$[L_y, L_x] = L_yL_x - L_xL_y = -(L_xL_y - L_yL_x) = -[L_x, L_y] = -i\hbar L_z$ (similar for other comp.)

$\Rightarrow$ using the definition of the Levi-Civita symbol: $\underline{\underline{[L_k, L_m] = +i\hbar\,\varepsilon_{kmn} L_n}}$

Note that here we use Einstein's summation rule: sum over same indices! (not yet important)

This problem can also be solved in a more formal, but purely algebraic (no differential operators) way (not required to obtain the points)

We already know: $\vec{L} = \vec{r} \wedge \vec{p}$, which can be written as $L_i = \underbrace{\varepsilon_{ijk} \cdot r_j \cdot p_k}$

sum over re-occuring indices

(Einstein's summation rule)

$$\Rightarrow [L_i, L_\ell] = [\varepsilon_{ijk} r_j p_k, \varepsilon_{\ell mn} r_m p_n] = \varepsilon_{ijk} \varepsilon_{\ell mn} [r_j p_k, r_m p_n]$$

From Algebra courses (or Wikipedia...) we know : $[ab, cd] = a[b,c]d + ac[b,d] + [a,c]db + c[a,d]b$

(this can be checked by direct inspection)

$$\Rightarrow \underline{[L_i, L_\ell]} = \varepsilon_{ijk} \varepsilon_{\ell mn} \cdot \{ r_j \underbrace{[p_k, r_m]}_{-i\hbar\delta_{km}} p_n + r_j r_m \underbrace{[p_k, p_n]}_{=0} + \underbrace{[r_j, r_m]}_{=0} p_n p_k + r_m \underbrace{[r_j, p_n]}_{+i\hbar\delta_{jn}} p_k \}$$

$$= + \underbrace{\varepsilon_{ijk} \varepsilon_{\ell mn}} \cdot i\hbar \{ -r_j p_n \cdot \delta_{km} + r_m p_k \cdot \delta_{jn} \}$$

Algebra course, wiki, or direct inspection

$$= i\hbar \{ \underbrace{\delta_{i\ell}}_{\substack{\text{drop this since} \\ i=\ell \text{ trivial!}}} \cancel{(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km})} - \delta_{im}(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}) + \delta_{in}(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}) \} \cdot \underbrace{\{ r_m p_k \delta_{jn} - r_j p_n \delta_{km} \}}$$

only 2 terms: $j=n$ and/or $k=m$
$\downarrow$ $\downarrow$
$\ell \neq j$ $\ell \neq k$
otherwise $\varepsilon = 0$ because of ident. indices.

from $j=n$, $\ell \neq j$

$$= i\hbar \{ \delta_{im} \delta_{k\ell} r_m p_k - \delta_{in} \delta_{j\ell} r_j p_n \}$$

$$= i\hbar \{ r_i p_\ell - r_\ell p_i \}$$

def. of ε

$$= i\hbar\, \varepsilon_{qi\ell} L_q \underset{\uparrow}{=} \underline{\underline{i\hbar\, \varepsilon_{i\ell q} \cdot L_q}}$$

cyclic permutation

2a

Assumption: we have an observable $\hat{A}$ with two eigenstates $|\alpha_1\rangle$ and $|\alpha_2\rangle$ with the corresponding eigenvalues a_1 and a_2.

$\Rightarrow$ A linear combination $\psi = \underset{\text{numbers}}{c_1} |\alpha_1\rangle + c_2 |\alpha_2\rangle$ results in

$$\hat{A}|\psi\rangle = c_1 \hat{A}|\alpha_1\rangle + c_2 \hat{A}|\alpha_2\rangle = c_1 a_1 |\alpha_1\rangle + c_2 a_2 |\alpha_2\rangle = a_1\left[c_1|\alpha_1\rangle + \frac{a_2}{a_1}\cdot c_2|\alpha_2\rangle\right]$$

i) if $a_1 = a_2$ (the eigenstates are degenerate) we obtain

$\hat{A}|\psi\rangle = a_1 [c_1|\alpha_1\rangle + c_2|\alpha_2\rangle] = a_1|\psi\rangle$, i.e. $|\psi\rangle$ is an eigenstate of $\hat{A}$.

ii) if $a_1 \neq a_2$ we obtain $\hat{A}\psi = a_1[c_1|\alpha_1\rangle + c_3|\alpha_2\rangle]$, i.e. a different linear combination than $|\psi\rangle$ $\quad c_3 = a_2/a_1$

$\Rightarrow$ no eigenstate.

b) Axioms for a scalar product (complex vector space):

$\hookrightarrow$ see "hermitian Sesquilinear form"

I) linear in first argument: $(x, ay+z) = a(x,y) + (x,z)$

II) semi-linear in second argument: $(ax+y, z) = a^*(x,z) + (y,z)$

III) hermitean: $(x,y) = (y,x)^*$

IV) positive: $(x,x) \geq 0 \quad \forall x$; $(x,x) = 0 \Leftrightarrow x = 0$

To show: if $(h, \hat{Q}h) = (\hat{Q}h, h) \quad \forall h \in \mathcal{H}$ (Hilbert space)

then $(f, \hat{Q}g) = (\hat{Q}f, g) \quad \forall f,g \in \mathcal{H}$

Solution: i) set $h = f+g$

$$\Rightarrow (f+g, \hat{Q}(f+g)) \underset{\hat{Q}\text{ linear and I) and II)}}{=} (f,\hat{Q}f) + (f,\hat{Q}g) + (g,\hat{Q}f) + (g,\hat{Q}g)$$

$$\underset{\text{from assumption}}{\overset{!}{=}} (\hat{Q}(f+g), f+g) \underset{\hat{Q}\text{ linear + I) and II)}}{=} (\hat{Q}f,f) + (\hat{Q}g,f) + (\hat{Q}f,g) + (\hat{Q}g,g)$$

$(f,\hat{Q}f)$ $\quad$ $(g,\hat{Q}g)$ – from assumption

or $\cancel{(f,\hat{Q}f)} + (f,\hat{Q}g) + (g,\hat{Q}f) + \cancel{(g,\hat{Q}g)} = \cancel{(\hat{Q}f,f)} + (\hat{Q}g,f) + (\hat{Q}f,g) + \cancel{(\hat{Q}g,g)}$ $\quad$ (Eq1)

ii) similarly set $h = f + ig$

$$\Rightarrow (f+ig, \hat{Q}(f+ig)) \underset{\substack{\uparrow \\ \hat{Q} \text{ and } (,) \\ \text{linear}}}{=} (f, \hat{Q}f) + (f, \hat{Q}(ig)) + (ig, \hat{Q}f) + (ig, \hat{Q}(ig))$$

$$= (f, \hat{Q}f) + i(f, \hat{Q}g) \underset{\substack{\uparrow \\ \text{semi-linear (II)}}}{-} i(g, \hat{Q}f) - \underbrace{i \cdot i}_{-1} (g, \hat{Q}g)$$

$$\underset{\substack{\uparrow \\ \text{assumption}}}{\overset{!}{=}} (\hat{Q}(f+ig), f+ig) = (\hat{Q}f, f) + i(\hat{Q}f, g) + (\hat{Q}(ig), f) + (\hat{Q}(ig), ig)$$

$$= (\hat{Q}f, f) + i(\hat{Q}f, g) \underset{\substack{\uparrow \\ \text{semi-lin (II)}}}{-} i(Qg, f) - \underbrace{i \cdot i}_{-1} (Qg, g)$$

or $\cancel{(f, \hat{Q}f)} + \cancel{i}(f, \hat{Q}g) - \cancel{i}(g, \hat{Q}f) + \cancel{(g, \hat{Q}g)} = \underbrace{(\hat{Q}f, f)}_{\cancel{(f, \hat{Q}f)}} + \cancel{i}(\hat{Q}f, g) - \cancel{i}(\hat{Q}g, f) + \underbrace{(\hat{Q}g, g)}_{\cancel{(g, \hat{Q}g)}}$ (Eq 2)

- from initial assumption

combining Eq 1 and Eq 2 results in

$\frac{1}{2}(\text{Eq 1} - \text{Eq 2}) \Rightarrow$ $\boxed{(g, \hat{Q}f) = (\hat{Q}g, f)}$

$\frac{1}{2}(\text{Eq 1} + \text{Eq 2}) \Rightarrow$ $\boxed{(f, \hat{Q}g) = (\hat{Q}f, g)}$