6. Harmonic Oscillator

This is a part that belongs to the chapter on harmonic oscillators. Due to time limits, I cannot discuss coherent states in the lectures. For those who are interested I have written this part. Please read it.

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6.1 Coherent States

For so called Fock-states, also called number-states (i.e. the eigenfunctions of the energy operator), the uncertainty product of position and momentum $\Delta x \cdot \Delta p = (n + 1/2)\hbar$. It is minimal in the ground state $n = 0$. We have also seen that the expectation values for position and momentum of all theses states are zero. Hence, these states display no movement in the sense of what we would expect to see in the classical motion of e.g. a pendulum. The oscillation of a pendulum is, if there is no friction, a state of constant energy. So there is seemingly a conflict between the quantum description and the classical description. However, one can show that quantum mechanics can describe the periodic motion surprisingly well, in fact with a wavepacket with minimal uncertainty. These states are superpositions of Fock-states and are known as coherent states. The expectation value for e.g. position of a coherent state is indeed a periodic motion. Even more so, the uncertainty in position and momentum of a coherent state is minimal, i.e. $\Delta x \cdot \Delta p = \hbar/2$.

Coherent states are eigenstates of the operator $\hat{a}_-$. Let us call these eigenstates ψ_α. The eigenvalue shall be c_α:

$$\hat{a}_- \psi_\alpha = c_\alpha \psi_\alpha \tag{1}$$

We will denote the Fock-states by ϕ_n:

$$\hat{H}\phi_n = E_n \phi_n = \hbar\omega \left(n + \frac{1}{2}\right) \phi_n \tag{2}$$

We also would like to recall the following set of equations for the raising and lowering operators $\hat{a}_+$, $\hat{a}_-$:

$$\begin{aligned}
\hat{x} &= \frac{x_0}{\sqrt{2}}(\hat{a}_+ + \hat{a}_-) \\
\hat{p} &= \frac{i\hbar}{\sqrt{2}x_0}(\hat{a}_+ - \hat{a}_-) \\
\hat{H} &= \hbar\omega \left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right) \\
\hat{H} &= \hbar\omega \left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) \\
1 &= [\hat{a}_-, \hat{a}_+]
\end{aligned} \tag{3}$$

The first exercise is to calculate the expectation values for position, momentum and its squares, i.e. $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p}^2 \rangle$ is the system is in one of the states ψ_α:

$$\begin{aligned}
\langle \hat{x} \rangle &= \frac{x_0}{\sqrt{2}} \langle \hat{a}_+ + \hat{a}_- \rangle = \frac{x_0}{\sqrt{2}} \left[(\psi_\alpha, \hat{a}_+ \psi_\alpha) + (\psi_\alpha, \hat{a}_- \psi_\alpha) \right] \\
&= \frac{x_0}{\sqrt{2}} \left[(\hat{a}_- \psi_\alpha, \psi_\alpha) + (\psi_\alpha, \hat{a}_- \psi_\alpha) \right] = \frac{x_0}{\sqrt{2}} (c_\alpha^\star + c_\alpha) = \frac{x_0}{\sqrt{2}} 2Re(c_\alpha)
\end{aligned} \tag{4}$$

$$\begin{aligned}
\langle \hat{p} \rangle &= i\frac{\hbar}{\sqrt{2}x_0} \langle \hat{a}_+ - \hat{a}_- \rangle = i\frac{\hbar}{\sqrt{2}x_0} \left[(\psi_\alpha, \hat{a}_+ \psi_\alpha) - (\psi_\alpha, \hat{a}_- \psi_\alpha) \right] \\
&= i\frac{\hbar}{\sqrt{2}x_0} \left[(\hat{a}_- \psi_\alpha, \psi_\alpha) - (\psi_\alpha, \hat{a}_- \psi_\alpha) \right] = i\frac{\hbar}{\sqrt{2}x_0} (c_\alpha^\star - c_\alpha) = \frac{\hbar}{\sqrt{2}x_0} 2Im(c_\alpha)
\end{aligned} \tag{5}$$

In the above equations we have also make use of the fact that the hermite conjugate operator of $\hat{a}_-$ is $\hat{a}_+$ and vice versa. Using the same procedure we can compute $\langle \hat{x}^2 \rangle$, $\langle \hat{p}^2 \rangle$, yielding:

$$\begin{aligned} \langle \hat{x}^2 \rangle &= \frac{x_0^2}{2} \left(1 + 2|c_\alpha|^2 + c_\alpha^2 + (c_\alpha^\star)^2\right) \\ \langle \hat{p}^2 \rangle &= \frac{\hbar^2}{2x_0^2} \left(1 + 2|c_\alpha|^2 - c_\alpha^2 - (c_\alpha^\star)^2\right) \end{aligned} \tag{6}$$

Based on equ. 4-6 we can calculate the standard deviations Δx and Δp. We obtain:

$$\Delta x = \frac{x_0}{\sqrt{2}} \qquad \Delta p = \frac{\hbar}{\sqrt{2} x_0} \tag{7}$$

This leads to the following minimal uncertainty product:

$$\Delta x \cdot \Delta p = \hbar/2 \tag{8}$$

Next we represent the state ψ_α in the basis given by the number states ϕ_n:

$$\psi_\alpha = \sum_{n=0}^{\infty} d_n \phi_n \tag{9}$$

According to the Fourier theorem, the coefficients d_n are obtained by the scalar product

$$d_n = (\phi_n, \psi_\alpha) = \frac{1}{\sqrt{n!}} (\hat{a}_+^n \phi_0, \psi_\alpha) \quad , \tag{10}$$

where ϕ_0 is the ground state. We have used the fact that the number state ϕ_n is obtained by a successive application of the raising operator $\hat{a}_+$. Continuing with equ. 10

$$d_n = \frac{1}{\sqrt{n!}} (\phi_0, \hat{a}_-^n \psi_\alpha) = \frac{c_\alpha^n}{\sqrt{n!}} (\phi_0, \psi_\alpha) \tag{11}$$

The last term $C = (\phi_0, \psi_\alpha)$ is a remaining constant, which needs to be determined through the normalization condition. The wavefunction ψ_α looks now as follows:

$$\psi_\alpha(x) = C \sum_{n=0}^{\infty} \frac{c_\alpha^n}{\sqrt{n!}} \phi_n(x) \tag{12}$$

The normalization condition $(\psi_\alpha, \psi_\alpha) = 1$ yields:

$$(\psi_\alpha, \psi_\alpha) = |C|^2 \sum_{n=0}^{\infty} \frac{|c_\alpha|^{2n}}{n!} = |C|^2 e^{|c_\alpha|^2} = 1 \tag{13}$$

Without loss of generality, let us chose the constant $C \in \mathbb{R}$, so that $C = exp(-|c_\alpha|^2/2)$. I now also add the time dependence, so that a coherent state can be decomposed as:

$$\begin{aligned} \psi_\alpha(x,t) &= e^{-|c_\alpha|^2/2} \sum_{n=0}^{\infty} \frac{c_\alpha^n}{\sqrt{n!}} \phi_n(x) e^{-iE_n t/\hbar} \\ \psi_\alpha(x,t) &= e^{-|c_\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(c_\alpha e^{-i\omega t})^n}{\sqrt{n!}} \phi_n(x) e^{-i\omega t/2} \end{aligned} \tag{14}$$

Let us now define a new number α as follows:

$$\alpha = c_\alpha e^{-i\omega t} \tag{15}$$

With this definition equation 18 can be written in a more compact form:

$$\psi_\alpha(x,t) = \left[e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \phi_n(x) \right] e^{-\omega t/2} \tag{16}$$

The interesting thing is that the part in the bracket has exactly the form of a coherent state, now with the eigenvalue α. The factor $exp(-\omega t/\hbar)$ is irrelevant as it does not lead to any observable effects.

The key point is that we have started with a coherent state at $t = 0$ with eigenvalue c_α As time evolves a coherent state remains a coherent state but with a new (time dependent) eigenvalue now given by $\alpha = c_\alpha exp(-i\omega t)$. This can also be expressed as:

$$\hat{a}_- \psi_\alpha(x, t = 0) = c_\alpha \psi_\alpha(x, t = 0) \qquad \hat{a}_- \psi_\alpha(x,t) = \alpha \psi_\alpha(x,t) \tag{17}$$

Now we can calculate the time-dependent expectation value of e.g. position:

$$\begin{aligned} \langle x \rangle = (\psi_\alpha, \hat{x}\psi_\alpha) &= \frac{x_0}{\sqrt{2}} \langle \hat{a}_+ + \hat{a}_- \rangle \\ &= \frac{x_0}{\sqrt{2}} \left[(\psi_\alpha, \hat{a}_+ \psi_\alpha) + (\psi_\alpha, \hat{a}_- \psi_\alpha) \right] \\ &= \frac{x_0}{\sqrt{2}} (\alpha^\star + \alpha) = \frac{x_0}{\sqrt{2}} \left[c_\alpha^\star e^{i\omega t} + c_\alpha e^{-i\omega t} \right] \\ &= \frac{2x_0|\alpha|}{\sqrt{2}} cos(\omega t + \delta_\alpha) \end{aligned} \tag{18}$$

where δ_α is the phase of c_α. Now clearly, we do have a harmonic oscillation! The amplitude is given by $\sqrt{2}|\alpha|x_0$. Hence, the magnitude of the eigenvalue corresponds to the amplitude of the coherent state. We also stress that there are infinite eigenstates of the operator $\hat{a}_-$. For any arbitrary value c_α a coherent state can be written down. In the limit of large amplitude, the state represents a classical motion quite accurately, because Δx is fixed to $x_0/\sqrt{2}$, while the amplitude grows. Hence, in this limit the wavepacket is narrow.

The last part of this exercise you should try to do yourself. The goal is to calculate the expectation values for a coherent state of the energy and the square of the energy from which one can derive the standard deviation, i.e. calculate $\langle H \rangle$, $\langle H^2 \rangle$ and ΔE. The results is:

$$\begin{aligned} \langle H \rangle &= \hbar\omega \left(|\alpha|^2 + \frac{1}{2} \right) \\ \langle H^2 \rangle &= (\hbar\omega)^2 (|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}) \\ \langle H^2 \rangle - (\langle H \rangle)^2 &= |\alpha|^2 (\hbar\omega)^2 \end{aligned} \tag{19}$$

Hence, we can also state that the energy is given by the amplitude (i.e. $|\alpha|$) squared and the energy uncertainty can be written for energies much larger than $\hbar\omega$ as

$$\Delta E \simeq \sqrt{\hbar\omega E} \quad \text{if} \quad E := \langle H \rangle \gg \hbar\omega \tag{20}$$

Let us emphasize once again, a coherent state is not an energy eigenstate. But for large amplitude (large energy) the energy uncertainty only grows with $\sqrt{E}$ so that the relative accuracy of the energy given by $\Delta E/E$ gets smaller and smaller as we pass over to the classical limit.