A1: Numerical simulation of the one-dimensional Schrödinger equation

There is a number of programs on the internet for calculating eigenstates and the time evolution of wavefunctions in one-dimensional potentials. Please go to http://www.falstad.com/qm1d/ and play with this applet (you can download the zip file and from https://java.com/en/download/ a Java engine.)

A2: Rectangular potential with one infinite wall

A particle of mass m moves in the one-dimensional potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ -V_0 & \text{for } 0 \leq x \leq L \\ 0 & \text{for } x > L, \end{cases}$$

with $V_0 > 0$.

(a) Scale the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)$$

by substituting the variables $x \mapsto yL$, $E \mapsto \varepsilon V_0$, $V(x) \mapsto v(y)V_0$. Show that this simplifies the Schrödinger equation to the dimensionless form

$$-\lambda\frac{\mathrm{d}^2\psi(y)}{\mathrm{d}y^2} + v(y)\psi(y) = \varepsilon\psi(y)$$

with the dimensionless parameter $\lambda = \frac{\hbar^2}{2mL^2V_0}$ that contains all specific parameters of the problem. Plot or draw schematically the resulting effective potential $v(y)$.

(b) Find the bound-state eigenenergies and corresponding wavefunctions for different regimes of λ. Note that bound states are characterized by energies $E < 0$ and thus $\varepsilon < 0$. For what values of λ are there no bound states? *Hint:* the transcendental equation for the eigenenergies can be solved graphically.

(c) Find the eigenstates of the free particles, characterized by eigenenergies $E \geq 0$. Why are all energy eigenvalues $E > 0$ allowed, *i.e.*, why is the energy spectrum continuous instead of discrete?

A3: Bound-state in Dirac-delta potential

Consider a particle of mass m moving in the attractive Dirac δ-potential $V(x) = -\alpha\delta(x)$. What is the probability that a measurement on its momentum (k-vector) yield a value greater than $k_0 = m\alpha/\hbar^2$?

(a) Use the real-space wavefunction $\psi(x)$ derived in the lecture:

$$\psi(x) = Be^{-\kappa|x|}.$$

Normalize the wavefunction first.

(b) Find-the momentum-space distribution by Fourier transform: You should obtain

$$\Phi(k) = \sqrt{\frac{2}{\pi}}\frac{\kappa^{3/2}}{\kappa^2 + k^2}$$

(c) Using the momentum-space distribution, calculate the probability. Use a table for the integral!