A1: Hydrogen Atom

The normalized hydrogen wavefunctions can be written as

$$\psi_{nlm}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right) Y_l^m(\theta,\phi),$$

where $a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ and $L^\alpha_n(x)$ and $Y_l^m(\theta,\phi)$ are:

$$L^\alpha_n(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

and

$$Y_l^m(\theta,\phi) = \frac{1}{2\pi} N_{lm} P_{lm}(\cos(\theta))\, e^{im\phi}$$

where

$$N_{lm} = \sqrt{\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}}$$

$$P_{lm}(x) = \left(1-x^2\right)^{|m|/2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{|m|} P_l(x)$$

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^{[l/2]} (-1)^k \frac{(2l-2k)!}{k!(l-k)!(l-2k)!} x^{l-2k}$$

and sign "[]" marks the integer part of the number (floor).

(a) Find explicit expressions for ψ_{210} and ψ_{211}.

For the following problems, consider an electron in the superposition state

$$\psi = \frac{1}{\sqrt{a^2+b^2+c^2}} (a\,\psi_{210} + b\,\psi_{211} + c\,\psi_{32-1}),$$

at $t=0$, for which a, b, and c can be assumed as real numbers.

(b) What results are possible for a measurement of the total energy? Give the probabilities of obtaining each result.

A2: Hydrogen Ground State

Consider the ground state wavefunction of the hydrogen atom.

(a) What is the most probable measurement result for the distance r between the electron and the nucleus? Hint: the answer is not 0! Why?

(b) What is the expectation value of r?

A3: Angular Momentum

The angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in Cartesian coordinates are

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Consider the state

$$\psi(x,y,z) = A\,\mathrm{e}^{-\alpha(x^2+y^2+z^2)} \left(\frac{x+iy}{\sqrt{x^2+y^2}}\right)^m$$

where $\alpha > 0$ and m is an integer.

(a) Determine the normalization constant A. Hint: $\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}$ for $a > 0$ (Gaussian integral).

(b) Calculate the expectation value of $\hat{L}_z$.

(c) Is the state ψ an eigenstate of $\hat{L}_z$? If so, what is its eigenvalue?

A4: Angular Momentum Consider the following Hamiltonian in three-dimensions with a potential that depends only on the distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(r),$$

where $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = -\hbar^2 \triangle$.

(a) Show in coordinate representation that the angular momentum operator in the z-direction

$$\hat{L}_z = xp_y - yp_x = -\imath\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right),$$

commutes with $\hat{p}^2$ (meaning that $[\hat{p}^2, \hat{L}_z] = 0$).