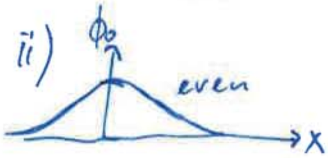
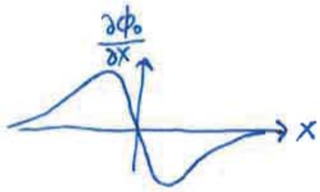
1a

Ground state $\Rightarrow \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega}{2\hbar}x^2}$

$\Rightarrow \phi_0(-x) = \phi_0(x)$ is an even function in x!

i) $\underline{\langle x \rangle} = \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{even}}}{\phi_0^*} \cdot \underset{\substack{\uparrow \\ \text{odd}}}{x} \cdot \underset{\substack{\uparrow \\ \text{even}}}{\phi}\, dx \; \underline{= 0}$ (could easily guess from symmetry)

ii)  $\Rightarrow$  $\Rightarrow \frac{d\phi_0}{\partial x}$ is odd in x

$\Rightarrow \underline{\langle p \rangle} = \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{even}}}{\phi_0^*} \Big(\underset{\substack{\uparrow \\ \text{odd}}}{-i\hbar \frac{d}{dx}\phi_0}\Big) dx \; \underline{= 0}$ (could easily be guessed by symmetry)

iii) $\underline{\langle x^2 \rangle} = \int_{-\infty}^{\infty} \phi_0^* x^2 \phi_0\, dx = \underbrace{2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \int_0^{\infty}}_{\text{for even functions}} x^2 \cdot e^{-\frac{m\omega}{\hbar}x^2} \underset{\substack{\uparrow \\ \text{Bronstein p. 1084} \\ \text{equation 26}}}{=} \cancel{2} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{\sqrt{\pi}}{{}_2\cancel{4}\left(\frac{m\omega}{\hbar}\right)^{3/2}} = \underline{\underline{\frac{\hbar}{2m\omega}}}$

iv) $\underline{\langle p^2 \rangle} = \int_{-\infty}^{\infty} \phi_0^* \left(-i\hbar \frac{d}{dx}\right)^2 \phi_0\, dx = -\hbar^2 \int_{-\infty}^{\infty} \phi_0^* \frac{d^2}{dx^2}\phi_0\, dx = -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \cdot \left(\frac{d^2}{dx^2} e^{-\frac{m\omega}{2\hbar}x^2}\right) dx$

$= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx}\left[-\frac{m\omega}{\cancel{2}\hbar} e^{-\frac{m\omega}{2\hbar}x^2} \cdot \cancel{2}x\right] dx$

$= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \left[-\frac{m\omega}{\hbar} e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x \cdot \left(-\frac{m\omega}{\cancel{2}\hbar} \cdot \cancel{2}x\right) e^{-\frac{m\omega}{2\hbar}x^2}\right] dx$

$= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \cdot e^{-\frac{m\omega}{2\hbar}x^2} \left[x^2 \cdot \frac{m\omega}{\hbar} - 1\right] dx$

$= -\hbar^2 \cdot \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar}\right)^{3/2} \cdot \Big[\underbrace{\frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx}_{\text{even} \Rightarrow \frac{\sqrt{\pi}}{2\left(\frac{m\omega}{\hbar}\right)^{3/2}}} - \underbrace{\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx}_{\text{even} \Rightarrow \frac{\sqrt{\pi}}{\sqrt{\frac{m\omega}{\hbar}}}}\Big]$

Bronstein p. 1084, equations 25 and 26

$= -\hbar^2 \cdot \cancel{\frac{1}{\sqrt{\pi}}} \cdot \left(\frac{m\omega}{\hbar}\right)^{3/2} \cdot \left[\cancel{\frac{m\omega}{\hbar}} \cdot \frac{\cancel{\sqrt{\pi}}}{2} \cdot \cancel{\frac{\hbar}{m\omega}} \cdot \sqrt{\frac{\hbar}{m\omega}} - \cancel{\sqrt{\pi}} \cdot \sqrt{\frac{\hbar}{m\omega}}\right]$

$= \frac{1}{2}\hbar^2 \frac{m\omega}{\hbar} = \underline{\underline{\frac{1}{2}\hbar m\omega}}$

v) $\Delta x = \sqrt{\langle x^2\rangle - \underbrace{\langle x\rangle^2}_{=0}} = \sqrt{\frac{\hbar}{2m\omega}}$

$\Delta p = \sqrt{\langle p^2\rangle - \underbrace{\langle p\rangle^2}_{=0}} = \sqrt{\frac{\hbar m\omega}{2}}$

$\Delta x \cdot \Delta p = \hbar/2$

Heisenberg uncertainty relation: $\Delta x \Delta p \geq \hbar/2$

$\Rightarrow$ ϕ_0 is a "minimum uncertainty" state.

1b

Some properties used for the solution:

$(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$ — operators

$(\psi, \hat{A}\varphi) = (\hat{A}^+\psi, \varphi)$ scalar product

$(\psi, \underset{\substack{|\\ \text{number}}}{\beta}\varphi) = (\beta^*\psi, \varphi) = \beta(\psi, \varphi)$

lowering and raising operators for the harmonic oscillator:

$(a_+)^+ = a_-$; $(a_-)^+ = a_+$

$[a_-, a_+] = a_-a_+ - a_+a_- = 1$

or $a_-a_+ = 1 + a_+a_-$

$\Rightarrow \underline{\underline{\langle H\rangle_\alpha}} = (\psi_\alpha, \hat{H}\psi_\alpha) = \frac{1}{2}\hbar\omega \underbrace{(\psi_\alpha, \psi_\alpha)}_{=1} + \hbar\omega(\psi_\alpha, a_+\underbrace{a_-\psi_\alpha}_{\alpha\cdot\psi_\alpha})$ (given by definition)

$= \frac{1}{2}\hbar\omega + \alpha\hbar\omega\underbrace{(\psi_\alpha, a_+\psi_\alpha)} = \underline{\underline{\hbar\omega(|\alpha|^2 + 1/2)}}$

$= ([a_+]^+\psi_\alpha, \psi_\alpha) = (a_-\psi_\alpha, \psi_\alpha) = \alpha^*\underbrace{(\psi_\alpha, \psi_\alpha)}_{=1}$

$\Rightarrow \langle H\rangle_\alpha^2 = (\hbar\omega)^2[|\alpha|^4 + |\alpha|^2 + 1/4]$

$\Rightarrow \langle H^2\rangle_\alpha = (\hbar\omega)^2(\psi_\alpha, [a_+a_- + 1/2]^+[a_+a_- + 1/2]\psi_\alpha)$

$= (\hbar\omega)^2(\psi_\alpha, [\underbrace{[a_+a_-]^+}_{=a_-^+a_+^+ = a_+a_-} + 1/2]\cdot[a_+a_- + 1/2]\psi_\alpha)$

$= (\hbar\omega)^2(\psi_\alpha, [a_+a_- + 1/2]\cdot[a_+a_- + 1/2]\psi_\alpha)$

$= (\hbar\omega)^2(\psi_\alpha, [\underset{(ii)}{a_+a_-a_+a_-} + 2\cdot 1/2\underset{(i)}{a_+a_-} + 1/4]\psi_\alpha)$

individual terms: (i) $(\psi_\alpha, a_+\underbrace{a_-\psi_\alpha}_{\alpha\cdot\psi_\alpha}) = \alpha\cdot(\psi_\alpha, a_+\psi_\alpha) = \alpha\cdot(\underbrace{(a_+)^+}_{a_-}\psi_\alpha, \psi_\alpha) = \alpha(\underbrace{a_-\psi_\alpha}_{\alpha\cdot\psi_\alpha}, \psi_\alpha)$

$= \alpha\cdot\alpha^*(\psi_\alpha, \psi_\alpha) = |\alpha|^2$

(ii) $(\Psi_\alpha, a_+ a_- a_+ \underbrace{a_-\Psi_\alpha}_{\alpha\Psi_\alpha}) = \alpha(\Psi_\alpha, a_+ \underbrace{a_- a_+}_{1+a_+a_-}\Psi_\alpha) = \alpha\underbrace{(\Psi_\alpha, a_+\Psi_\alpha)}_{}+\alpha(\Psi_\alpha, a_+^2 \underbrace{a_-\Psi_\alpha}_{\alpha\cdot\Psi_\alpha})$

see above as
$((a_+)^\dagger\Psi_\alpha, \Psi_\alpha) = (\alpha\Psi_\alpha, \Psi_\alpha)$
$= \alpha^*(\Psi_\alpha, \Psi_\alpha) = \alpha^*$

$= \alpha^*\alpha \cdot \alpha\cdot\alpha\cdot(\Psi_\alpha, a_+^2\Psi_\alpha) \quad = |\alpha|^2 + \alpha\cdot\alpha\cdot\alpha^*\underbrace{(\Psi_\alpha, a^+\Psi_\alpha)}_{\alpha^*}$

$= |\alpha|^2 + |\alpha|^4$

$$\Rightarrow \underline{\underline{\langle H^2\rangle_\alpha = \left[|\alpha|^4 + 2|\alpha|^2 + 1/4\right]\cdot(\hbar\omega)^2}}$$

$$\Rightarrow \underline{\Delta H} = \sqrt{\langle H^2\rangle_\alpha - \langle H\rangle_\alpha^2}$$

$$= \hbar\omega\sqrt{|\alpha|^4 + 2|\alpha|^2 + 1/4 - |\alpha|^4 - |\alpha|^2 - 1/4} = \underline{\underline{\hbar\omega\cdot|\alpha|}}$$

1c) (continued)

$\Rightarrow$ In contrast to Δx (variance in position) and Δp (variance in momentum), ΔH <u>does</u> depend on α.

For comparison (see lecture notes): one can use the same way to calculate Δx and Δp, with $\hat{x} = \frac{\lambda}{\sqrt{2}}(a_- + a_+)$

$\hat{p} = \frac{-i\hbar}{\sqrt{2}}(a_- - a_+)$, $\lambda = \sqrt{\frac{\hbar}{m\omega}}$.

One then finds $\Delta x\cdot\Delta p = \hbar/2$, i.e. these „coherent states" have the momentum and position with minimal uncertainty (Heisenberg inequality).