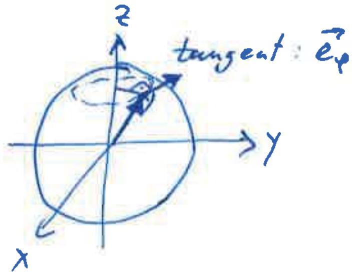
2a) classical torque: $\underline{\underline{\vec{T} = \vec{\mu} \wedge \vec{B}}}$

explicitly: in cartesian coordinates we have $\vec{\mu} = \mu \begin{pmatrix} \sin(\theta)\cdot\cos(\varphi) \\ \sin(\theta)\cdot\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$ $\Big]$ $\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$

$$\Rightarrow \vec{T} = \vec{\mu} \wedge \vec{B} = \mu \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \mu\cdot B \cdot \begin{pmatrix} \sin(\theta)\sin(\varphi) \\ -\sin(\theta)\cos(\varphi) \\ 0 \end{pmatrix}$$

$$= \mu \cdot B \cdot \sin(\theta) \cdot \underbrace{\begin{pmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix}}_{= -\vec{e}_\varphi}$$

$= -\vec{e}_\varphi$: unit vector for changes in φ in spherical coordinates



$$\Rightarrow \underline{\underline{\vec{T} = -\mu B \cdot \sin(\theta)\cdot \vec{e}_\varphi}}$$

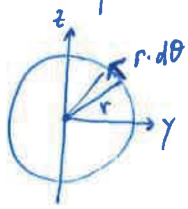
b) goal: i) derive differential equation for $\vec{L}(t)$, i.e. obtain $\frac{\partial \vec{L}}{\partial t}$

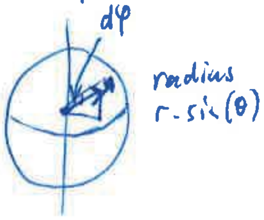
ii) solve differential equation

i) In polar coordinates the change in a vector $\vec{s}$ when the coordinates are changed by dr (radius), $d\theta$ and $d\varphi$ (angles) is

$$d\vec{s} = \vec{e}_r \cdot dr + \vec{e}_\theta \cdot r \cdot d\theta + \vec{e}_\varphi \cdot r \cdot \sin(\theta)\cdot d\varphi$$

radial unit vector





$$\Rightarrow \frac{d\vec{s}}{dt} = \vec{e}_r \cdot \frac{dr}{dt} + \vec{e}_\theta \cdot r \cdot \frac{d\theta}{dt} + \vec{e}_\varphi \cdot r \cdot \sin(\theta)\cdot \frac{d\varphi}{dt}$$

or for $\vec{L}$: $\underline{\underline{\frac{d\vec{L}}{dt}}} = \vec{e}_r \cdot \frac{dL}{dt} + \vec{e}_\theta \cdot L \cdot \frac{d\theta}{dt} + \vec{e}_\varphi \cdot L \cdot \sin(\theta)\cdot\frac{d\varphi}{dt}$

$$\overset{!}{=} \underset{\text{by definition}}{\vec{T}} \overset{a)}{=} \underline{\underline{-\mu B \cdot \sin(\theta)\cdot \vec{e}_\varphi}}$$

(Differential equation for $\vec{L}$)

with $L = -\frac{2m}{g_L \cdot e}\,\mu$

ii) solve differential equation(s): • radial component: $\frac{dL}{dt} \overset{!}{=} 0 \Rightarrow \underline{L = \text{const.}}$

• $\vec{e}_\theta$-component: $L \cdot \frac{d\theta}{dt} \overset{!}{=} 0$, $L \neq 0 \Rightarrow \underline{\underline{\theta = \text{const.}}}$

• $\vec{e}_\varphi$ - component: $L \cdot \cancel{\sin(\theta)} \, \frac{d\varphi}{dt} \stackrel{!}{=} -\mu B \cdot \cancel{\sin(\theta)}$

$$\frac{d\varphi}{dt} = -\frac{\mu B}{L} = -\frac{\cancel{\mu} B}{-\frac{2m}{g_L e}\cdot \cancel{\mu}} = \frac{g_L e}{2m}\cdot B$$

$$\Rightarrow \underline{\underline{\varphi(t) = \frac{g_L e}{2m} B \cdot t}}$$

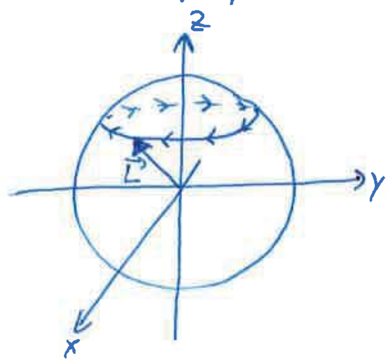
c) Since L, θ are constant and $\varphi \propto t$ we obtain

$$\vec{L}(t) = L \cdot \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}, \text{ i.e. } L_z = \text{const.}, \quad \begin{aligned} L_x &= L\cdot\sin(\theta)\cdot\cos(\varphi) = L_\parallel \cdot \cos(\varphi(t)) \\ L_y &= L\cdot\sin(\theta)\cdot\sin(\varphi) = L_\parallel \cdot \sin(\varphi(t)) \end{aligned}$$

which describes a circle with radius $L_\parallel = L \cdot \sin(\theta)$.

Rotation frequency: full period $\hat{=}$ $\varphi(\Delta t) = 2\pi = \frac{g_L e}{2m} B \cdot \Delta t$

$$\Rightarrow \underline{\underline{\omega = \frac{2\pi}{\Delta t} = \frac{g_L e}{2m} \cdot B}} \quad \text{(Larmor frequency)}$$



Larmor - precession:

$\chi_+^z = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_-^z = \begin{pmatrix}0\\1\end{pmatrix}$ eigenstates of σ_z

$H = -\gamma \vec{B}\vec{S}$ $\qquad E_+ = -\frac{\hbar}{2}\gamma B \quad E_- = \frac{\hbar}{2}\gamma B$

time evolution

usually: $\psi(t) = \sum_n c_n e^{-iE_n t/\hbar}\varphi_n$

$\chi(t) = a\,\chi_+^z e^{-iE_+t/\hbar} + b\,\chi_-^z e^{-i\underset{E_-}{\overset{?}{E_2}}t/\hbar} \qquad \chi(0) = \begin{pmatrix}a\\b\end{pmatrix}$

notation: $b := \sin(\alpha/2) \quad a := \cos(\alpha/2) \rightarrow |a|^2+|b|^2 = 1$ normalized

$$\chi(t) = \cos\left(\frac{\alpha}{2}\right)\chi_+^z e^{-iE_+t/\hbar} + \sin\left(\frac{\alpha}{2}\right)\chi_-^z e^{-iE_-t/\hbar} =$$

$$= \begin{pmatrix}\cos(\alpha/2)\, e^{-iE_+t/\hbar}\\ \sin(\alpha/2)\, e^{-iE_-t/\hbar}\end{pmatrix} = \begin{pmatrix}\cos(\alpha/2)\, e^{i\gamma Bt/2}\\ \sin(\alpha/2)\, e^{-i\gamma Bt/2}\end{pmatrix}$$

$$\langle S_x\rangle = \chi^* S_x \chi = \chi^*(t)\cdot\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\, e^{i\gamma Bt/2}\\ \sin(\alpha/2)\, e^{-i\gamma Bt/2}\end{pmatrix} =$$

$$= \frac{\hbar}{2}\left(\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma Bt/2}, \sin\left(\frac{\alpha}{2}\right)e^{i\gamma Bt/2}\right)\begin{pmatrix}\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma Bt/2}\\ \cos\left(\frac{\alpha}{2}\right)e^{i\gamma Bt/2}\end{pmatrix} =$$

$$= \frac{\hbar}{2}\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma Bt} + \frac{\hbar}{2}\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{i\gamma Bt/2} =$$

$2\sin x\cdot\cos x = \sin(2x)$

$$= \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\Big[\underbrace{e^{-i\gamma Bt} + e^{i\gamma Bt}}_{2\cos(\gamma Bt)}\Big]\frac{\hbar}{2} = \sin(\alpha)\cos(\gamma Bt)\cdot\frac{\hbar}{2}$$

similarly

$$\langle S_y\rangle = -\frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t)$$

using σ_y and σ_z

$$\langle S_z\rangle = \frac{\hbar}{2}\cos(\alpha)$$